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TRIGONOMETRY,

PLANE AND SPHERICAL;

WITH

THE CONSTRUCTION AND APPLICATION

OF

LOGARITHMS.

BY THOMAS SIMPSON, F. R. S.

1710-1761

WITH

AN APPENDIX

ON

SPHERICAL PROJECTIONS.

PHILADELPHIA:
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1810.

DISTRICT OF PENNSYLVANIA, TO WIT:

(L. S.) BE IT REMEMBERED, That on the fourth day of December, in the thirty-fifth year of the Independence of the United States of America, A. D. 1810,

KIMBER AND CONRAD,

of the said district, have deposited in this office the title of a book, the right whereof they claim as proprietors, in the words following, to wit:

Trigonometry, Plane and Spherical; with the Construction and Application of Logarithms. By Thomas Simpson, F. R. S. With an Appendix on Spherical Projections.

In conformity to the act of the congress of the United States, entitled, "An act for the encouragement of learning, by securing the copies of maps, charts, and books, to the authors and proprietors of such copies during the times therein mentioned." And also to the act, entitled, "An act supplementary to an act, entitled, "An act for the encouragement of learning, by securing the copies of maps, charts, and books, to the authors and proprietors of such copies during the times therein mentioned," and extending the benefits thereof to the arts of designing, engraving, and etching historical and other prints."

D. CALDWELL,
Clerk of the District of Pennsylvania.

PREFACE.

THE work, of which an American edition is now offered to public acceptance, needs no laboured encomium, as it is the production of an author whose peculiar facility in combining the elegance of the profound geometer with the perspicuity of the practical teacher, has justly established his character with the mathematical world.

The tract having been originally designed as an appendix to a treatise on geometry published by the same writer, it was thought expedient to adapt the references to a different work, more generally studied in our schools. They are accordingly suited to Playfair's Geometry, but they will mostly apply to Simson's translation of Euclid's Elements. Some cases, which were omitted by T. Simpson, have been supplied; particularly the use and demonstration of Napier's circular parts.

The appendix, now subjoined, it is hoped, will prove acceptable, and render the work more extensively useful. The propositions are principally extracted from the writings of Emerson; but many of the demonstrations are either partly or entirely new.

E. L.

Newgarden, 11 mo. 24, 1810.

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Jno. Marton Jno
1818

PLANE TRIGONOMETRY.

DEFINITIONS.

1. **PLANE** Trigonometry is the art whereby, having given any three parts of a plane triangle (except the three angles), the rest are determined. In order to which, it is not only requisite that the peripheries of circles, but also certain right lines in and about the circle be supposed divided into some assigned number of equal parts.

2. The periphery of every circle is supposed to be divided into 360 equal parts, called degrees; and each degree into 60 equal parts, called minutes; and each minute into 60 equal parts, called seconds, or second minutes, &c.

3. Any part AB (fig. 1.) of the periphery of the circle is called an arch, and is said to be the measure of the angle ACB at the centre, which it subtends.

Note. *The degrees, minutes, seconds, &c. contained in any arch, or angle, are written in this manner,* 50° 18′ 35″, *which signifies that the given arch, or angle, contains* 50 *degrees,* 18 *minutes, and* 35 *seconds.*

4. The difference of any arch from 90° (or a quadrant) is called its complement; and its difference from 180° (or a semicircle) its supplement.

5. A chord, or subtense, is a right line drawn from one extremity of an arch to the other: thus the right line BE is the chord, or subtense, of the arch BAE or BDE.

6. The sine, or right sine, of an arch is a right line drawn from one extremity of the arch, perpendicular to the diameter passing through the other extremity. Thus BF is the sine of the arch AB or DB.

7. The versed sine of an arch is the part of the diameter intercepted between the sine and the periphery. Thus AF is the versed sine of AB; and DF of DB.

8. The co-sine of an arch is the part of the diameter intercepted between the centre and sine, and is equal to the sine of the complement of that arch. Thus CF is the co-sine of the arch AB, and is equal to BI, the sine of its complement HB.

9. The tangent of an arch is a right line touching the circle in one extremity of that arch, produced from thence till it meets a right line passing through the centre and the other extremity. Thus AG is the tangent of the arch AB.

10. The secant of an arch is a right line reaching, without the circle, from the centre to the extremity of the tangent. Thus CG is the secant of AB.

11. The co-tangent and co-secant of an arch are the tangent and secant of the complement of that arch. Thus HK and CK are the co-tangent and co-secant of AB.

12. A trigonometrical canon is a table exhibiting the length of the sine, tangent, and secant to every degree and minute of the quadrant, with respect to the radius; which is supposed unity, and conceived to be divided into 10,000 or more decimal parts. By the help of this table, and the doctrine of similar triangles, the whole business of trigonometry is performed; which I shall now proceed to show. But, first of all, it will be proper to observe, that the sine of any arch A*b* greater than 90° is equal to the sine of another arch AB as much below 90°; and that its co-sine C*f*, tangent A*g*, and secant C*g*, are also respectively equal to the co-sine, tangent, and secant of its supplement AB; but only are negative, or fall on contrary sides of the points C and A, from whence they have their origin. All which is manifest from the definitions.

THEOREM I.

In any right-angled plane triangle ABC (fig. 2.), *it will be as the hypothenuse is to the perpendicular, so is the radius (of the table) to the sine of the angle at the base.*

For, let AE or AF be the radius to which the table of sines, &c. is adapted, and ED the sine of the angle A or arch EF (*Vid. Def.* 3. *and* 6.); then, because of the similar triangles ACB and AED, it will be AC : BC : : AE : ED (*by* 4. 6.). *Q. E. D.*

Thus, if AC = ,75, and BC = ,45; then it will be, ,75 : ,45 : : 1 (radius) : the sine of A = 6; which, in the table, answers to 36° 52′, the measure or value of A.

THEOREM II.

In any right-angled plane triangle ABC (fig. 2.), *it will be, as the base* AB *is to the perpendicular* BC, *so is the radius (of the table) to the tangent of the angle at the base.*

For, let AE or AF be the radius of the table, or canon, and FG the tangent of the angle A, or arch EF (*Vid. Def.* 3. *and* 9.); then, by reason of the similarity of the triangles ABC, AFG, it will be AB : BC : : AF : FG. *Q. E. D.*

Thus, let AB = ,8, and BC = ,5; then we shall have ,8 : ,5 : : 1 (radius) : tangent A = ,625; whence A itself is found, by the canon, to be 32° 00′.

THEOREM III.

In every plane triangle ABC (fig. 3.), *it will be, as any one side is to the sine of its opposite angle, so is any other side to the sine of its opposite angle.*

For, take CF = AB, and upon AC let fall the perpendiculars BD and FE; which will be the sines of the angles A and C to the equal radii AB and CF. Now the triangles CBD, CFE being similar, we have CB : BD (*sin.* A) : : CF (AB) : FE (*sin.* C). *Q. E. D*

THEOREM IV.

As the base of any plane triangle ABC (fig. 4.) *is to the sum of the two sides, so is the difference of the sides to twice the distance* DE *of the perpendicular from the middle of the base.*

From the vertex C, with the greater side AC, describe a circle, and produce AB and CB to the circumference; then it is evident that BF is the sum, and BH the difference of the sides; also, since AB + BG = 2AD (3. 3.) = 2AE + 2ED = AB + 2ED; BG = 2ED. Join AF, HG; then the angle AFB = BGH; FAB = BHG (21. 3.), and ABF = HBG; therefore AB : BF (AC + BC) : : BH (AC — BC) : BG (2ED), by 4. 6. *Q. E. D.*

THEOREM V.

In any plane triangle, it will be, as the sum of any two sides is to their difference, so is the tangent of half the sum of the two opposite angles, to the tangent of half their difference.

For, let ABC (fig. 5.) be the triangle, and AB, AC the two proposed sides; and from the centre A, with the radius AB, let a circle be described, intersecting CA, produced, in D and F; so that CF may express the sum, and CD the difference of the sides AC and AB: join F, B and B, D, and draw DE parallel to FB, meeting BC in E.

Then, because $2ADB = ADB + ABD$ (*by* 5. 1.) $= C + ABC$ (*by* 32. 1.) it is plain that ADB is equal to half the sum of the angles opposite to the sides proposed. Moreover, since $ABC = ABD\ (ADB) + DBC$, and $C = ADB - DBC$ (*by* 32. 1.) it is plain that $ABC - C$ is $= 2DBC$; or that DBC is equal to half the difference of the same angles.

Now, because of the parallel lines BF and ED, it will be CF : CD : : BF : DE; but BF and DE, because DBF and BDE are right angles (*by* 31. 3. *and* 29. 1.), will be tangents of the foresaid angles FDB (ADB) and DBE (DBC) to the radius BD. *Q. E. D.*

COROLLARY.

Hence, in two triangles ABC and A*b*C, having two sides equal, each to each, it will be (*by equality*), as tang. $\frac{AbC + ACb}{2}$: tang. $\frac{AbC - ACb}{2}$: : tang. $\frac{ABC + ACB}{2}$: tang. $\frac{ABC - ACB}{2}$. But, if CA*b* be supposed a right angle, then will $AbC + ACb$ also = a right angle (*by* 32. 1.), and the tangent of $\frac{AbC + ACb}{2}$ = radius*. Therefore in this case our proportion will become,

Radius : tang. $\frac{AbC - ACb}{2}$ $(= AbC - 45°)$: : tang. $\frac{ABC + ACB}{2}$: tang. $\frac{ABC - ACB}{2}$. Which gives the following *theorem*, for finding the angles opposite to any

* See Prop. 1. Cor. 5.

two proposed sides; the included angle, and the sides themselves, being known.

As the less of the proposed sides (A*b or* AB) *is to the greater* (AC), *so is radius to the tangent of an angle* (A*b*C, see Theor. 2.) *And as radius to the tangent of the excess of this angle above* 45°, *so is the tangent of half the sum of the required angles to the tangent of half their difference**.

* This theorem, though it requires two proportions, is commonly used by astronomers in determining the elongation and parallaxes of the planets (being best adapted to logarithms); for which reason it is here given.

The solution of the cases of right-angled plane triangles.

Figure 6.

Case.	Given	Sought	Solution.
1	The hyp. AC and the angles	One leg BC	As radius is to the sine of A, so is the hyp. AC to the leg BC (*by Theor. I.*)
2	The hyp. AC and one leg BC	The angles	As AC : BC : : radius : sin. A (*Theor. I.*), whose complement is the angle C.
3	The hyp. AC and one leg BC	The other leg AB	Let the angles be found, *by Case* 2. and then the required leg AB *by Case* 1.
4	The angles and one leg BC	The hyp. AC	As sine A : radius : : the leg BC : the hyp. AC (*Theor. I.*)
5	The angles and one leg BC	The other leg AB	As sine A : BC : : sine C : AB (*by Theor. III.*) Or, as radius : tang. C : : BC : AB (*by Theor. II.*)
6	The two legs AB and BC	The angles	As AB : BC : : radius : tang. A (*by Theor. II.*), whose complement is the angle C.
7	The two legs AB and BC	The hyp. AC	Let the angles be found, *by Case* 6. and then the hyp. AC, *by Case* 4.

The solution of the cases of oblique plane triangles.

Figures 7, 8.

Case.	Given	Sought	Solution.
1	The angles and one side AB	Either of the other sides BC	As sine C : AB : : sine A : BC (*by Theor. III.*)
2	Two sides AB, BC and an ang. C op. to one of them	The other angles A and ABC	As AB : sin. C : : BC : sin. A (*by Theor. III.*) which added to C, and the sum subtracted from 180, gives the other angle ABC.
3	Two sides AB, BC, and an opp. angle C	The other side AC	Let the angle ABC be found, by the preceding case, and then it will be, sin. C : AB : : sin. ABC : AC (*by Theor. III.*)
4	Two sides AC, AB and the included angle A	The other angles C and ABC	As sum of AB and AC : their dif. : : tang. of half the sum of ABC and C : tang. of half their diff. (*by Theor. V.*) which added to, and subtracted from, the half sum, gives the two angles.
5	Two sides AC, AB and the incl. ∢A	The other side BC	Let the angles be found by the last case, and then BC, *by case* 1.
6	All the three sides	An angle, suppose A	Let fall a perp. BD opp. to the req. angle : then (*by Theor. IV.*) as AC : sum of AB and BC : : their dif. : twice dist. DG of the perp. from the middle of the base; whence AD being also known the angle A will be found by *Case* 2 of right angles.

Note.—The 2d and 3d cases are ambiguous, or admit of two different answers each, when the side AB opposite the given angle C (*see fig.* 8.) is less than the given side BC, adjacent to it (except the angle found be exactly a right one): for then another right line B*a*, equal to BA, may be drawn from B to a point in the base, somewhere between C and the perpendicular BD, and therefore the angle found by the proportion AB (*a*B) : sin. C : : BC : sin. A (*or of* C*a*B), may, it is evident, be either the acute angle A, or the obtuse one C*a*B, which is its supplement, the sines of both being exactly the same.

Having laid down the method of resolving the different cases of plane triangles, by a table of signs and tangents; I shall here show the manner of constructing such a table, as the foundation upon which the whole doctrine is grounded; in order to which, it will be requisite to premise the following propositions.

PROPOSITION I.

The sine of an arch being given, to find its co-sine, versed sine, tangent, co-tangent, secant, and co-secant.

Let AE (fig. 9.) be the proposed arch, EF its sine, CF its co-sine, AF its versed sine, AT its tangent, CT its secant, DH its co-tangent, and CH its co-secant.

Then (*by* 47. 1.) we have CF = the square root of $CE^2 - EF^2$; whence, not only the co-sine CF, but also the versed sine AF, will be known. Then, because of the similar triangles CFE, CAT, and CDH, it will be (*by* 4. 6.)

1. CF : FE : : CA : AT; whence the tangent is known.

2. CF : CE (CA) : : CA : CT; whence the secant is known.

3. EF : CF : : CD : DH; whence the co-tangent is known.

4. EF : EC (CD) : : CD : CH; whence the co-secant is known.

Hence it appears,

1. That the tangent is a fourth proportional to the co-sine, the sine, and radius.

2. That the secant is a third proportional to the co-sine and radius.

3. That the co-tangent is a fourth proportional to the sine, co-sine, and radius.

4. And that the co-secant is a third proportional to the sine and radius.

5. It appears, moreover (because AT : AC : : CD (AC) : DH), that the rectangle of the tangent and co-tangent is equal to the square of the radius (*by* 17. 6.): whence it likewise follows, that the tangent of half a right angle is equal to the radius; and that the co-tangents of any two different arches, represented by P and Q, are to one another, inversely as the tangents of the same arches: for, since tang. P × co-tang. P = squ. rad. = tang. Q × co-

tang. Q; therefore will co-tang. P : co-tang. Q : : tang. Q : tang. P; or as co-tang. P : tang. Q : : co-tang. Q : tang. P (*by* 16. 5.).

PROP. II.

If there be three equidifferent arches AB, AC, AD (fig. 10.), *it will be, as radius is to the co-sine of their common difference* BC *or* CD, *so is the sine* CF *of the mean, to half the sum of the sines* BE + DG, *of the two extremes: and, as radius to the sine of the common difference, so is the co-sine* FO *of the mean to half the difference of the sines of the two extremes.*

For let BD be drawn, intersecting the radius OC in *m*; also draw *mn* parallel to CF, meeting AO in *n*; and BH and *mv*, parallel to AO, meeting DG in H and *v*.

Then, the arches BC and CD being equal to each other (*by hypothesis*), OC is not only perpendicular to the chord BD, but also bisects it (*by* 3. 3.), and therefore B*m* (or D*m*) will be the sine of BC (or DC), and O*m* its co-sine: moreover *mn*, being an arithmetical mean between the sines BE, DG of the two extremes (because B*m* = D*m*) is therefore equal to half their sum, and D*v* equal to half their difference. But, because of the similar triangles OCF, O*mn*, and D*vm*,

It will be $\left\{\begin{array}{l} \text{OC} : \text{O}m :: \text{CF} : mn \\ \text{OC} : \text{D}m :: \text{FO} : \text{D}v \end{array}\right\}$. *Q. E. D.*

COROLLARY I.

Because of the foregoing proportions, we have m $\left(\frac{DG + BE}{2}\right) = \frac{Om \times CF}{OC}$*, and $Dm - \left(\frac{DG - BE}{2}\right) = \frac{Dm \times FO}{OC}$; and therefore $DG + BE = \frac{2Om \times CF}{OC}$, and $DG - BE = \frac{2Dm \times FO}{OC}$.

COROLLARY II.

Hence, if the mean arch AC be supposed that of 60°; then, OF being the co-sine of 60°, = sine 30° = $\frac{1}{2}$ chord of 60° = $\frac{1}{2}$ OC, it is manifest that DG — BE will, in this case, be barely = Dm; and consequently DG = Dm + BE. From whence, and the preceding corollary, we have these two useful theorems.

1. *If the sine of the mean of three equidifferent arches (supposing radius unity) be multiplied by twice the co-sine of the common difference, and the sine of either extreme be subtracted from the product, the remainder will be the sine of the other extreme.*

* *Note.* $\frac{Om \times CF}{OC}$, taken in a *geometrical sense*, denotes a fourth proportional to OC, Om, and CF; but, *arithmetically*, it signifies the quantity arising by dividing the product of the measures of Om and CF by that of OC. Understand the like of others.

2. *The sine of any arch, above* 60 *degrees, is equal to the sine of another arch, as much below* 60°, *together with the sine of its excess above* 60°.

PROP. III.

To find the sine of a very small arch; suppose that of 15′.

It is found, in p. 181 of the Elements*, that the length of the chord of $\frac{1}{384}$ of the semi-periphery is expressed by ,00818121 (radius being unity); therefore, as the chords of very small arches are to each other nearly as the arches themselves (*vid. p.* 181), we shall have, as $\frac{1}{384} : \frac{1}{360}$: : ,00818121 : ,008726624, the chord of $\frac{1}{360}$, or half a degree; whose half, or ,004363312, is therefore the sine of 15′, very nearly.

From whence the sine of any inferior arch may be found by bare proportion. Thus, if the sine of 1′ be required, it will be 15′ : 1′ : : ,004363312 : ,000290888, the sine of the arch of one minute, nearly.

But if you would have the sine of 1′ more exactly determined (from which the sines of other arches may be derived with the same degree of exactness); then let the operations in p. 181 be continued to 11 bisections, and a greater number of decimals be taken; by which means you will get the

* The Elements of Geometry published by the same author are here referred to. The same conclusion is easily derived from the scholium to Prop. 11, book 1, Supplement to Playfair's Geometry.

chord of $\frac{1}{8144}$ part of the semi-periphery to what accuracy you please: then, by proceeding as above (by finding the sine of 15′), the sine of 1′ will also be obtained to a very great degree of exactness.

PROP. IV.

To show the manner of constructing the trigonometrical canon.

First, find the sine of an arch of one minute, by the preceding Prop. and then its co-sine by Prop. I. which let be denoted by C; then (*by Theor. I. p.* 17.) we shall have

$2C \times \text{sine } 1' - \text{sine } 0' = \text{sine } 2'$.
$2C \times \text{sine } 2' - \text{sine } 1' = \text{sine } 3'$.
$2C \times \text{sine } 3' - \text{sine } 2' = \text{sine } 4'$.
$2C \times \text{sine } 4' - \text{sine } 3' = \text{sine } 5'$.

And thus are the sines of 6′, 7′, 8′, &c. successively derived from each other.

The sines of every degree and minute, up to 60°, being thus found; those of above 60° will be had by addition only (*from Theor.* 2. *p.* 18.); then, the sines being all known, the tangents and secants will likewise become known, *by Prop.* 1.

Note. If the sine of every 5th minute, *only*, be computed according to the foregoing method, the sines of all the intermediate arches may be had from thence, by *barely* taking the proportional parts of the differences, and that so near as

to give the [illegible] places true in each number; which is sufficiently exact for all common purposes.

SCHOLIUM.

Although what has been hitherto laid down for constructing the trigonometrical canon, is abundantly sufficient for that purpose, and is also very easily demonstrated; yet, as the first sine, from whence the rest are all derived, must be carried on to a great number of places, to render the numerous deductions from it but tolerably exact (because in every operation the error is multiplied), I shall here subjoin a different method, which will be found to have the advantage, not only in that, but in many other respects.

First, then, from the co-sine of 15°, which is $= \frac{1}{2}\sqrt{2+\sqrt{3}}$* $= ,965925826$, &c.†, and the sine of 18°, which is $= \frac{1}{2}\sqrt{\frac{5}{4}} - \frac{1}{2} = ,309017$, &c. (equal to half the side of a decagon inscribed in the circle), let the co-sine of 3°, the difference

* The side of a hexagon being equal to the radius of its circumscribing circle (15. 4.), it follows that the sine of 30° (radius being unity) $= \frac{1}{2}$, and its co-sine $(\sqrt{1-\frac{1}{4}}) = \frac{1}{2}\sqrt{3}$; whence the versed sine of 150° $(1 +$ co-s. of $30°) = \frac{2+\sqrt{3}}{2}$; consequently the co-sine of $15° = \sqrt{\frac{2+\sqrt{3}}{4}}$ (*prop.* 1, *p.* 54) $= \frac{1}{2}\sqrt{2+\sqrt{3}}$.

† It appears (*by* 11. 2. *and* 10. 4.) that if the side of an isosceles triangle, each of whose angles at the base is double the third angle, be $= 1$, the base $= \sqrt{\frac{5}{4}} - \frac{1}{2}$; and if from the vertex of such a triangle a perpendicular be demitted on the base, the semibase is manifestly the sine of half the vertical angle, that is, the sine of 18° $= \frac{1}{2}\sqrt{\frac{5}{4}} - \frac{1}{4}$.

between 18° and 15° be found*; from which the co-sine of 45′ will be had, by two bisections only: whence the sines of all the arches in the progression 1° 30′, 2° 15′, 3° 00′, 3° 45′, &c. may be determined (*by Theor.* 1. *p.* 17), and that to any assigned degree of exactness.

The sines of all the terms of the progression 45′, 1° 30′, 2° 15′, &c. up to 60°, being thus derived, the next thing is to find, by the help of these, the sines of all the intermediate arches, to every single minute.

This, if you desire no more than the 4 or 5 first places of each (which is exact enough where nothing less than degrees and minutes is regarded), may be effected by barely taking the proportional parts of the differences.

But if a greater degree of accuracy be insisted on, and you would have a table carried on to 7 or 8 places, each number (which is sufficient to give the value of an angle to seconds, and even to thirds, in most cases), then the operation may be as follows:

1°. Multiply the sum of the sines of any two adjacent terms of the progression 45′, 1° 30′, 2° 15′, 3° 00′, 3° 45′, &c. (betwixt which you would find all the intermediate sines) by the fraction ,0000000423, for a first product; and this, again, by 22, for a second product; to which last let $\frac{1}{47}$ of the difference of the two proposed sines (or extremes) be added, and the sum will be the excess of the first of the intermediate sines above the lesser extreme.

* *Note.* The co-sine of the difference of two arches (supposing radius unity) is found by adding the product of their sines to that of their co-sines; as is hereafter demonstrated.

2°. From this excess let the first product be continually subtracted; that is, first, from the excess itself; then from the remainder; then from the last remainder, and so on 44 times.

3°. To the lesser extreme add the forementioned excess; and to the sum add the first remainder; to this sum add the next remainder, and so on continually: then the several sums thus arising will respectively exhibit the sines of all the intermediate arches, to every single minute, exclusive of the last; which, if the work be right, will agree with the greater extreme itself, and therefore will be of use in proving the operation.

But to illustrate the matter more clearly, let it be proposed to find the sines of all the intermediate arches between 3° 00′ and 3° 45′ to every single minute, those of the extremes being given, from the foregoing method, equal to ,05233595 and ,06540312 respectively. Here, the sum of the sines of the extremes being multiplied by ,0000000423, the first product will be ,00000000498, &c. or ,0000000050, *nearly* (which is sufficiently exact for the present purpose); and this, again, multiplied by 22, gives ,00000011 for a 2d product; which added to ,0002903815, $\frac{1}{47}$ part of the difference of the two given extremes, will be ,0002904915, the excess of the sine of 3° 01′ above that of 3° 00′. From whence, by proceeding according to the 2d and 3d rules, the sines of all the other intermediate arches are had, by addition and subtraction only. See the operation.

,000290	4915	excess	,05233595	sine 3° 0′
,000000	0050		,0002904915	
	4865	1st. rem.	,0526264415	sine 3° 1′
	50		2904865	
	4815	2d. rem.	,0529169280	sine 3° 2′
	50		2904815	
	4765	3d. rem.	,0532074095	sine 3° 3′
	50		2904765	
	4715	4th. rem.	,0534978860	sine 3° 4′
	50		2904715	
	4665	5th. rem.	,0537883575	sine 3° 5′
	50		2904665	
	4615	6th. rem.	,0540788240	sine 3° 6′
	50		2904615	
	4565	7th. rem.	,0543692855	sine 3° 7′
	50		2904565	
	4515	8th. rem.	,0546597420	sine 3° 8′
	50		2904515	
	4465	9th. rem.	,0549501935	sine 3° 9′
	50		2904465	
	4415	10th. rem.	,0552406400	sine 3° 10′
	&c.		&c.	

Again, as a second example, let it be required to find the sines of all the arches, to every minute, between 59° 15′ and 60° 00′; those of the two extremes being first found, by the preceding method. In this case, the two extremes being ,85940641 and ,86602540, their sum will be = 1,72543, &c. and their difference = ,00661899; whereof the former, multiplied by ,0000000423 (*see the rule*), gives ,00000007298,

&c. or ,0000000730, nearly, for the first product (which is exact enough for our purpose); therefore the 2d product, or ,0000000730 × 22, will be ,0000016060; which, added to $\frac{1}{45}$ of the difference, gives ,0001486947; from whence the operation will be as follows:

,00014	86947 excess	,85940641	sine 59° 15′
,00000	00730 1st. prod.	0001486947	
	86217 1st. rem.	,8595551047	sine 59° 16′
	730	1486217	
	85487 2d. rem.	,8597037264	sine 59° 17′
	730	1485487	
	84757 3d. rem.	,8598522751	sine 59° 18′
	730	1484757	
	84027 4th. rem.	,8600007508	sine 59° 19′
	730	1484027	
	83297 5th. rem.	,8601491535	sine 59° 20′
	730	1483297	
	82567 6th. rem.	,8602974832	sine 59° 21′
		1482567	
		,8604457399	sine 59° 22′
	&c.	&c.	

After the same manner the sines of all the intermediate arches between any other two proposed extremes may be derived, even up to 90 degrees; but those of above 60° are best found from those below, as has been shown elsewhere.

The reasons upon which the foregoing operations are founded depend upon principles too foreign from the main

design of this treatise to be explained here (even would room permit); however, as to the correctness and utility of the method itself, I will venture to affirm, that, whoever has the inclination, either to calculate new tables, or to examine those already extant, will not find one quarter of the trouble this way as he unavoidably must according to the common methods.

SPHERICAL TRIGONOMETRY.

DEFINITIONS.

1. A great circle of a sphere is a section of the sphere by a plane passing through the centre.

2. The axis of a circle is a right line passing through the centre of the sphere, perpendicular to the plane of the circle: and the two points, where the axis intersects the surface of the sphere, are called the poles of the circle.

3. A spherical angle is the inclination of two great circles.

4. A spherical triangle is a part of the surface of the sphere included by the arches of three great circles; which arches are called the sides of the triangle.

5. If through the poles A and F (fig. 11.) of two great circles DF and DA, standing at right angles, two other great circles ACE and FCB be conceived to pass, and thereby form two spherical triangles ABC and FCE, the latter of the triangles so formed is said to be the complement of the former; and *vice versa*.

COROLLARIES.

1. It is manifest (*from Def.* 1.) that the section of two great circles (as it passes through the centre) will be a diameter of the sphere; and consequently, that their peripheries will always intersect each other in two points at the distance of a semicircle, or 180 degrees.

2. It also appears (*from Def.* 2.) that all great circles, passing through the pole of a given circle, cut that circle at right angles; because they pass through or coincide with the axis, which is perpendicular to it.

3. It follows morever, that the periphery of a great circle is everywhere 90 degrees distant from its pole; and that the measure of a spherical angle CAD*(fig. 12.) is an arch of a great circle intercepted by the two circles ACB, ADB forming that angle, and whose pole is the angular point A. For let the diameter AB be the intersection of the great circles ADB and ACB (*see Corol.* 1.), and let the plane, or great circle, DEC be conceived perpendicular to that diameter, intersecting the surface of the sphere in the arch CD; then it is manifest that AD = BD = 90°, and AC = BC = 90° (*Corol.* 1.), and that CD is the measure of the angle DEC (*or* CAD), the inclination of the two proposed circles.

4. Hence it is also manifest, that the angles B and E, (fig. 11.) of the complemental triangles ABC and FCE, are both right angles; and that CE is the complement of AC, CF of

*Note. Although a spherical angle is, properly, the inclination of two great circles, yet it is commonly expressed by the inclination of their peripheries at the point where they intersect each other.

BC, BD (*or the angle* F) of AB, and EF of ED (*or the angle* A).

THEOREM I.

In any right-angled spherical triangle it will be, as radius is to the sine of the angle at the base, so is the sine of the hypothenuse to the sine of the perpendicular; and as radius to the co-sine of the angle at the base, so is the tangent of the hypothenuse to the tangent of the base.

DEMONSTRATION.

Let ADL and AEL (fig. 13.) be two great circles of the sphere intersecting each other in the diameter AL, making an angle DOE, measured by the arch ED; the plane DOE being supposed perpendicular to the diameter AL, at the centre O.

Let AB be the base of the proposed triangle, BC the perpendicular, AC the hypothenuse, and BAC (*or* DAE = DE = DOE) the angle at the base: moreover, let CG be the sine of the hypothenuse, AK its tangent, AI the tangent of the base, CH the sine of the perpendicular, and EF the sine of the angle at the base; and let I, K and G, H be joined.

Because CH is perpendicular to the plane of the base (*or paper*), it is evident, that the plane GHC will be perpendicular to the plane of the base, and likewise perpendicular to the diameter AL, because GC, being the sine of AC, is perpendicular to AL. Moreover, since both the planes OIK and AIK are perpendicular to the plane of the base (*or paper*), their intersection IK will also be perpendicular to it, and consequently the angle AIK a right angle. Therefore, seeing

the angles OFE, GHC, and AIK are all right angles, and that the planes of the three triangles OFE, GHC, and AIK are all perpendicular to the diameter AL, we shall, *by similar triangles*,

$$\text{have} \begin{cases} OE : EF :: GC : CH \\ OE : OF :: AK : AI \end{cases}$$

that is, Radius : sine of EOF (*or* BAC) : : sine of AC : sine of BC.
Radius : co-sine of EOF (*or* BAC) : : tang. AC : tang. AB. *Q. E. D.*

COROLLARY I.

Hence it follows, that the sines of the angles of any oblique spherical triangles ADC are to one another, directly, as the sines of the opposite sides.

For let BC (fig. 14.) be perpendicular to AD;

$$\begin{matrix}\text{then} \\ \text{since}\end{matrix} \begin{cases} \text{radius : sine A : : sine AC : sine BC} \\ \text{radius : sine D : : sine DC : sine BC} \end{cases} \textit{by the former part of the theorem;}$$

we shall have, sine A × sine AC (= radius × sine BC) = sine D × sine DC (*by* 16. 6.), and consequently sine A : sine D : : sine DC : sine AC; or sine A : sine DC : : sine D : sine AC.

COROLLARY 2.

It follows, moreover, that, in right-angled spherical triangles ABC, DBC, having one leg BC common, the tangents of the hypothenuses are to each other, inversely, as the co-sines of the adjacent angles.

$$\begin{matrix}\text{For} \\ \text{since}\end{matrix} \begin{cases} \text{radius : co-sine ACB : : tan. AC : tan. BC} \\ \text{radius : co-sine DCB : : tan. DC : tan. BC} \end{cases}$$

by the latter part of the theorem; we shall (*by arguing as above*) have co-sine ACB : co-sine DCB : : tang. DC : tang. AC.

THEOREM II.

In any right-angled spherical triangle (ABC) (fig. 11.) *it will be, as radius is to the co-sine of one leg, so is the co-sine of the other leg to the co-sine of the hypothenuse.*

DEMONSTRATION.

Let CEF be the complemental triangle to ABC, according to what has been already specified; then it will be (*by Theor.* 1. *Case* 1.)

Radius : sine F : : sine CF : sine CE ; that is,

Radius : co-sine BA : : co-sine CB : co-sine AC (*see Cor.* 4. *p.* 27.) *Q. E. D.*

COROLLARY.

Hence, if two right-angled spherical triangles ABC, CBD (fig. 14.) have the same perpendicular BC, the co-sines of their hypothenuses will be to each other, directly, as the co-sines of their bases.

For $\left\{\begin{array}{l}\text{rad : co-sin. BC : : co-sin. AB : co-sine AC,}\\ \text{since rad : co-sin. BC : : co-sin. DB : co-sine DC,}\end{array}\right.$

therefore, by equality and permutation, co-sine AB : co-sine DB : : co-sine AC : co-sine DC.

THEOREM III.

In any right-angled spherical triangle (ABC) (fig. 11.) *it will be, as radius is to the sine of either angle, so is the co-sine of the adjacent leg to the co-sine of the opposite angle.*

DEMONSTRATION.

Let CEF be as in the preceding proposition; then, *by Theor.* 1. *Case* 1. it will be, radius : sine C : : sine CF : sine EF; that is, radius : sine C : : co-sine BC : co-sine A. *Q. E. D.*

COROLLARY.

Hence, in right-angled spherical triangles ABC, CBD, having the same perpendicular BC (fig. 14.), the co-sines of the angles at the base will be to each other, directly, as the sines of the vertical angles:

For $\left\{\begin{array}{l}\text{radius : sine BCA : : co-sine CB : co-sine A,}\\ \text{radius : sine BCD : : co-sine CB : co-sine D,}\end{array}\right.$
since

therefore, *by equality and permutation,*

Co-sine A : co-sine D : : sine BCA : sine BCD.

THEOREM IV.

In any right-angled spherical triangle (ABC) (fig. 11.) *it will be, as radius is to the sine of the base, so is the tangent of the angle at the base to the tangent of the perpendicular.*

For, supposing CEF as before,

It will be, as radius : co-sine of F : : tang. CF : tang. FE (*by the latter part of Theor.* 1.); that is, radius : sine AB : :

co-tang. BC : co-tang. A : : tang. A : tang. BC (*by Corol. 5. p.* 15.) *Q. E. D.*

COROLLARY.

Hence it follows, that, in right-angled spherical triangles ABC, DBC (fig. 14.), having the same perpendicular BC, the sines of the bases will be to each other, *inversely*, as the tangents of the angles at the bases:

$$\left.\begin{array}{l}\text{For}\\ \text{since}\end{array}\right\}\begin{array}{l}\text{radius : sine AB : : tang. A : tang. BC}\\ \text{radius : sine DB : : tang. D : tang. BC}\end{array}\Bigg\}$$

we shall (*by reasoning as in Cor.* 1. *Theor.* 1.) have

Sine AB : sine DB : : tang. D : tang. A.

THEOREM V.

In any right-angled spherical triangle it will be, as radius is to the co-sine of the hypothenuse, so is the tangent of either angle to the co-tangent of the other angle.

For, CEF (fig. 11.) *being as in the last*, it will be, as radius : sine CE : : tang. C : tang. EF (*by Theorem* 4.) ; that is, radius : co-sine AC : : tang. C : co-tang. A. *Q. E. D.*

LEMMA.

As the sum of the sines of two unequal arches is to their difference, so is the tangent of half the sum of those arches to the tangent of half their difference: and, as the sum of the co-sines is to their difference, so is the co-tangent of half

the sum of the arches to the tangent of half the difference of the same arches.

For, let AB and AC (fig. 15.) be the two proposed arches, and let BG and CH be their sines, and OG and OH their co-sines: moreover, let the arch BC be equally divided in D, so that CD may be half the difference, and AD half the sum, of AB and AC: let the radii OD and OC be drawn, and also the chord CB, meeting OE in E and OA produced in P; draw ES parallel to AO, meeting CH in S, and EF and OK perpendicular to AO, and let the latter meet EC produced in I; lastly, draw QDK perpendicular to OD, meeting OA, OC, and QI produced in Q, L, and K.

Because CD = BD, it is manifest that OD is not only perpendicular to the chord BC, but bisects it in E; whence, also, EF bisects HG; and therefore CH + BG = 2EF, and CH — BG = 2CS; also OG + OH = 2OF, and OG — OH = 2HF: but 2EF (CH + BG) : 2CS (CH — BG) : : EF : CS : : EP : EC (*by* 4. 6.) : : DQ (the tangent of AD) : DL (the tangent of DC). And 2OF (OG + OH) : 2HF (OG — OH) : : EI : EC : : DK (the co-tang. of AD) : DL (the tang. DC). *Q. E. D.*

THEOREM VI.

In any spherical triangle ABC (fig. 16.) *it will be, as the co-tangent of half the sum of the two sides is to the tangent of half their difference, so is the co-tangent of half the base to the tangent of the distance* (DE) *of the perpendicular from the middle of the base.*

DEMONSTRATION.

Since co-sine AC : co-sine BC : : co-sine AD : co-sine BD (*by Cor. to Theor.* 2.); therefore, by composition and division, co-sine AC + co-sine BC : co-sine AC — co-sine BC : : co-sine AD + co-sine BD : co-sine AD — co-sine BD. But (*by the preceding lemma*) co-sine AC + co-sine BC : co-sine AC — co-sine BC : : co-tang. $\frac{AC + BC}{2}$: tang. $\frac{AC - BC}{2}$; and co-sine AD + co-sine BD : co-sine AD — co-sine BD : : co-tang. of AE $\left(\frac{AD + BD}{2}\right)$: tang. DE $\left(\frac{AD - BD}{2}\right)$; whence, by equality, co-tang. $\frac{AC + BC}{2}$: tang. $\frac{AC - BC}{2}$: : co-tang. AE : tang. DE.

COROLLARY.

Since the last proportion, by permutation, becomes co-tang. $\frac{AC + BC}{2}$: co-tang. AE : : tang. $\frac{AC - BC}{2}$: tang. DE, and it is proved, in *p.* 15, that the tangents of any two arches are, inversely, as their co-tangents; it follows, therefore, that tangent AE : tangent $\frac{AC + BC}{2}$: : tang. $\frac{AC - BC}{2}$: tang. DE; or, that *the tangent of half the base is to the tangent of half the sum of the sides, as the tangent of half the difference of the sides to the tangent of the distance of the perpendicular from the middle of the base.*

THEOREM VII.

In any spherical triangle ABC (fig. 16.), *it will be, as the co-tangent of half the sum of the angles at the base is to the tangent of half their difference, so is the tangent of half the vertical angle to the tangent of the angle which the perpendicular* CD *makes with the line* CF *bisecting the vertical angle.*

DEMONSTRATION.

It will be (*by Cor. to Theor.* 3.), co-sine A : co-sine B : : sine ACD : sine BCD; and therefore, co-sine A + co-sine B : co-sine A — co-sine B : : sine ACD + sine BCD : sine ACD — sine BCD. But (*by the lemma*) co-tang. $\frac{B+A}{2}$: tang. $\frac{B-A}{2}$: : (co-sine A + co-sine B : co-sine A — co-sine B : : sine ACD + sine BCD : sine ACD — sine BCD) tang. ACF : tang. DCF. *Q. E. D.*

The solution of the cases of right-angled spherical triangles.

Figure 17.

Case.	Given	Sought	Solution.
1	The hyp. AC and one angle A	The opposite leg BC	As radius : sine hyp. AC : : sine A : sine BC (*by the former part of Theor.* 1.)
2	The hyp. AC and one angle A	The adjacent leg AB	As radius : co-sine of A : : tang. AC : tang. AB (*by the latter part of Theor.* 1.)
3	The hyp. AC and one angle A	The other angle C	As radius : co-sine of AC : : tang. A : co-tang. C (*by Theor.* 5.)
4	The hyp. AC and one leg AB	The other leg BC	As co-sine AB : radius : : co-sine AC : co-sine BC (*by Theor.* 2.)
5	The hyp. AC and one leg AB	The opposite angle C	As sine AC : radius : : sine AB : sine C (*by the former part of Theor.* 1.)
6	The hyp. AC and one leg AB	The adjacent angle A	As tang. AC : tang. AB : : radius : co-sine A (*by Theor.* 1.)
7	One leg AB and the adjacent angle A	The other leg BC	As radius : sine AB : : tangent A : tangent BC (*by Theor.* 4.)
8	One leg AB and the adjacent angle A	The opposite angle C	As radius : sine A : : co-sine of AB : co-sine of C (*by Theor.* 3.)

Case.	Given	Sought	Solution.
9	One leg AB and the adjacent angle A	The hyp. AC	As co-sine of A : radius : : tang. AB : tang. AC (*by Theor.* 1.)
10	One leg BC and the opposite angle A	The other leg AB	As tang. A : tang. BC : : radius : sine AB (*by Theor.* 4.)
11	One leg BC and the opposite angle A	The adjacent angle C	As co-sine BC : radius : : co-sine of A : sine C (*by Theor.* 3.)
12	One leg BC and the opposite angle A	The hyp. AC	As sine A : sine BC : : radius : sine AC (*by Theor.* 1.)
13	Both legs AB and BC	The hyp. AC	As radius : co-sine AB : : co-sine BC : co-sine AC (*by Theor.* 2.)
14	Both legs AB and BC	An angle, suppose A	As sine AB : radius : : tang. BC : tang. A (*by Theor.* 4.)
15	Both angles A and C	A leg, suppose AB	As sine A : co-sine C : : radius : co-sine AB (*by Theor.* 3.)
16	Both angles A and C	The hyp. AC	As tang. A : co-tang. C : : radius : co-sine AC (*by Theor.* 5.)

Note. The 10th, 11th, and 12th cases are ambiguous; since it cannot be determined, by the data, whether AB, C, and AC be greater or less than 90 degrees each.

In every right-angled spherical triangle there are five parts, susceptible of different values, viz. the legs, the oblique angles, and the hypothenuse. Now the legs themselves, the complements of the oblique angles, and the complement of the hypothenuse, are called the *circular parts*, as they follow each other round the triangle. Of these, any one being assumed as the *middle part*, those nearest to it are called the *adjacent extremes*, and the other two *opposite extremes*.

Baron Napier, the inventor of logarithms, discovered two theorems, by which all the cases of right-angled spherical triangles may be solved; and which, being easily remembered, are frequently used in practice.

Theor. 1. The rectangle of radius and the sine of the middle part is equal to the rectangle of the tangents of the adjacent extremes.

Theor. 2. The rectangle of the radius and sine of the middle part is equal to the rectangle of the co-sines of the opposite extremes.

The following table exhibits the different cases and the correspondent equations arising from these theorems.

Figure 17.

Cases.	Mid part.	Adjacent extremes	Opposite extremes.	Equations.
1	AB	Comp. A BC	Comp C Comp. AC	r. × s. AB = { cot. A × tan. BC s. C × s. AC
2	BC	Comp. C AB	Comp A Comp. AC	r. × s BC = { cot. C × tan. AB s. A × s. AC
3	Comp. A	Comp. AC AB	Comp. C BC	r. × cos. A = { cot. AC × tan. AB s. C × cos. BC
4	Comp. C	Comp. AC BC	Comp. A AB	r. × cos. C = { cot. AC × tan. BC s. A × cos. AB
5	Comp. AC	Comp. A Comp. C	AB BC	r. × cos. AC = { cot. A × cot. C cos. AB × cos. BC

DEMONSTRATION.

Cases 1, 2. Making AB the middle part.

As rad. : sine AB : : tang. A : tang. BC (*Theor.* 4.); whence, by inversion and alternation, sine AB : tang. BC : : rad. : tang. A : : co-tang. A : radius (*Cor.* 5. *Prop.* 1.); consequently r. × sine AB = co-tang. A × tang. BC.

Again, r. : sine C : : sine AC : sine AB (*Theor.* 1.); whence r. × sine AB = sine C × sine AC.

In the same manner the theorems may be proved when BC is made middle part.

Cases 3, 4. Making the complement of A middle part.

As rad. : co-sine A : : tang. AC : tang. AB (*Theor.* 1.); and, by inversion and alternation, co-sine A : tang. AB : :

rad. : tang. AC : : co-tang. AC : rad. (*Cor.* 5. *Prop.* 1.); whence rad. × co-sine A = co-tang. AC × tang. AB.

Again, rad. : sine C : : co-sine BC : co-sine A (*Theor.* 3.); whence rad. × co-sine A = sine C × co-sine BC.

In like manner the theorems may be proved when the complement of C is middle part.

Case 5. Complement of AC middle part.

As rad. : co-sine AC : : tang. A : co-tang. C (*Theor.* 5.); whence, by inversion and alternation, co-sine AC : co-tang. C : : rad. : tang. A : : co-tang. A : rad. (*Cor.* 5. *Prop.* 1.); consequently rad. × co-sine AC = co-tang. A × co-tang. C.

Again, rad. : co-sine AB : : co-sine BC : co-sine AC (*Theor.* 2.), and therefore rad. × co-sine AC = co-sine AB × co-sine BC.

When any two of these circular parts are given to find a third, we must assume such one middle part as will make the other two either both adjacent or both opposite extremes.

The solution of the cases of oblique spherical triangles.

Figures 16, 18.

Case.	Given	Sought	Solution.
1	Two sides AC, BC, and an angle A opposite to one of them	The angle B opposite to the other	As sine BC : sine A : : sine AC : sine B (*by Cor.* 1. *to Theor.* 1.) *Note.* This case is ambiguous when BC is less than AC; since it cannot be determined from the data, whether B be acute or obtuse.
2	Two sides AC, BC, and an angle A opposite to one of them	The included angle ACB	Upon AB produced (if need be) let fall the perpendicular CD; then (*by Theor.* 5.) rad. : co-sine AC : : tang. A : co-tang. ACD; but (*by Cor.* 2. *to Theor.* 1.) as tang BC : tang. AC : : co-sine ACD : co-sine BCD. Whence ACB = ACD ± BCD is known.
3	Two sides AC, BC, and an angle opposite to one of them	The other side AB	As rad. : co-sine A : : tang. AC : tang. AD (*by Theor.* 1.); and (*by Cor. to Theor.* 2.) as co-sine AC : co-sine BC : : co-sine AD : co-sine BD. *Note.* This and the last case are both ambiguous when the first is so.
4	Two sides AC, AB, and the included angle A	The other side BC	As rad. : co-sine A : : tang AC : tang. AD (*by Theor.* 1.); whence BD is also known; then (*by Cor. to Theor.* 2.) as co-sine AD : co-sine BD : : co-sine AC : co-sine BC.
5	Two sides AC, AB, and the included angle A	Either of the other angles, suppose B	As rad. : co-sine A : : tang. AC : tang. AD (*by Theor.* 1.); whence BD is known; then (*by Cor. to Theor.* 4) as sine BD : sine AD : : tang. A : tang. B.
6	Two angles A, ACB, and the side AC betwixt them	The other angle B	As rad. : co sine AC : : tang. A : co-tang. ACD (*by Theor.* 5.); whence BCD is also known; then (*by Cor. to Theor.* 3.) as sine ACD : sine BCD : : co-sine A : co-sine B.
7	Two angles A, ACB, and the side AC betwixt them	Either of the other sides, suppose BC	As rad. : co-sine AC : : tang. A : co-tang. ACD (*by Theor.* 5.); whence BCD is also known; then as co-sine of BCD : co sine ACD : : tang. AC : tang. BC (*by Cor.* 2. *to Theor.* 1.)

Case.	Given	Sought	Solution.
8	Two angles A, B, and a side AC opposite to one of them	The side BC opposite the other	As sine B : sine AC : : sine A : sine BC (*by Cor.* 1. *to Theor.* 1.)
9	Two angles A, B, and a side AC opposite to one of them	The side AB betwixt them	As rad. : co-sine A : : tang. AC : tang. AD (*by Theor.* 1.); and as tang. B : tang. A : : sine AD : sine BD (*by Cor. to Theor.* 4.); whence AB is also known.
10	Two angles A, B, and a side AC opposite to one of them	The other angle ACB	As rad. : co-sine AC : : tang. A : co-tang. ACD (*by Theor.* 5.); and as co-sine A : co-sine B : : sine ACD : sine BCD (*by Cor. to Theor.* 3.); whence ACB is also known.
11	All the three sides AB, AC, and BC	An angle; suppose A	As tang. $\frac{1}{2}$AB : tang. $\frac{AC + BC}{2}$: : tang. $\frac{AC - BC}{2}$: tang. DE, the distance of the perpendicular from the middle of the base (*by Cor. to Theor.* 6.); whence AD is known; then, as tang. AC : tang. AD : : rad. : co-sine A (*by Theor.* 1.)
12	All the three angles A, B, and ACB	A side, suppose AC	As co-tang. $\frac{ABC+A}{2}$: tang. $\frac{ABC-A}{2}$: : tang. $\frac{ACB}{2}$: tang. of the angle included by the perpendicular and a line bisecting the vertical angle; whence ACD is also known; then (*by Theor.* 5.) tang. A : co-tang. ACD : : rad. : co-sine AC.

Note. In letting fall your perpendicular, let it always be from the end of a given side and opposite to a given angle.

Of the Nature and Construction of Logarithms, with their application to the doctrine of Triangles.

AS the business of trigonometry is wonderfully facilitated by the application of logarithms; which are a set of artificial numbers, so proportioned among themselves and adapted to the natural numbers 2, 3, 4, 5, &c. as to perform the same things by addition and subtraction, *only*, as these do by multiplication and division: I shall here, for the sake of the young beginner (for whom this small tract is chiefly intended), add a few pages upon this subject. But, first of all, it will be necessary to premise something, in general, with regard to the indices of a geometrical progression, whereof logarithms are a particular species.

Let, therefore, 1, a, a^2, a^3, a^4, a^5, a^6, a^7, &c. be a geometrical progression, whose first term is unity, and common ratio any given quantity a. Then it is manifest,

1. *That the sum of the indices of any two terms of the progression is equal to the index of the product of those terms.* Thus $2 + 3$ (5) is = the index of $a^2 \times a^3$, or a^5; and $3 + 4$ (= 7) is = the index of $a^3 \times a^4$, or a^7. This is universally demonstrated in p. 19 of my book of *Algebra*.

2. *That the difference of the indices of any two terms of the progression is equal to the index of the quotient of one of them divided by the other.* Thus $5 - 3$ is = the index of

$\frac{a^5}{a^3}$ or a^2. Which is only the converse of the preceding article.

3. *That the product of the index of any term by a given number* (n) *is equal to the index of the power whose exponent is the said number* (n). Thus 2×3 (6) is = the index of a^2 raised to the third power (or a^6). This is proved in p. 43, and also follows from article 1.

4. *That the quotient of the index of any term of the progression by a given number* (n) *is equal to the index of the root of that term defined by the same number* (n). Thus $\frac{6}{3}$ (2) is = the index of (a^2) the cube root of a^6. Which is only the converse of the last article.

These are the properties of the indices of a geometrical progression; which being universally true, let the common ratio be now supposed indefinitely near to that of equality, or the excess of a above unity, indefinitely little; so that some term or other of the progression 1, a, a^2, a^3, a^4, a^5, &c. may be equal to, or coincide with each term of the series of natural numbers, 2, 3, 4, 5, 6, 7, &c. *Then are the indices of those terms called logarithms of the numbers to which the terms themselves are equal.* Thus, if $a^m = 2$, and $a^n = 3$, then will m and n be logarithms of the numbers 2 and 3 respectively.

Hence it is evident, that what has been above specified, in relation to the properties of the indices of powers, is equally true in the logarithms of numbers; since logarithms are nothing more than the indices of such powers as agree in

value with those numbers. Thus, for instance, if the logarithms of 2 and 3 be denoted by m and n; that is, if $a^m = 2$, and $a^n = 3$, then will the logarithm of 6 (the product of 2 and 3) be equal to $m + n$ (agreable to article 1.); because $2 \times 3\ (6) = a^m \times a^n = a^{m+n}$.

But we must now observe, that there are various forms or species of logarithms; because it is evident that what has been hitherto said, in respect to the properties of indices, holds equally true in relation to any equimultiples, or like parts, of them; which have, manifestly, the same properties and proportions, with regard to each other, as the indices themselves. But the most simple kind of all is *Napier's*, otherwise called the *hyperbolical.*

The hyperbolical logarithm of any number is the index of that term of the logarithmic progression agreeing with the proposed number, multiplied by the excess of the common ratio above unity.

Thus, if e be an indefinitely small quantity, the hyperbolic logarithm of the natural number agreeing with any term $\overline{1+e}|^n$ of the logarithmic progression 1, $\overline{1+e}$, $\overline{1+e}|^2$, $\overline{1+e}|^3$, $\overline{1+e}|^4$, &c. will be expressed by ne.

PROPOSITION I.

The hyperbolic logarithm (L) *of a number being given, to find the number itself, answering thereto.*

Let $\overline{1+e}]^n$ be that term of the logarithmic progression $1, \overline{1+e}]^1, \overline{1+e}]^2, \overline{1+e}]^3, \overline{1+e}]^4$, &c. which is equal to the required number (N). Then, because $\overline{1+e}]^n$ is, universally, $= 1 + ne + n \cdot \frac{n-1}{2} \cdot e^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot e^3$, &c. we shall, also, have $1 + ne + n \cdot \frac{n-1}{2} \cdot e^2 \cdot + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot e^3$, &c. $=$ N. But, because n (from the nature of logarithms) is here supposed indefinitely great, it is evident, first, that the numbers connected to it by the sign — may be rejected as far as any assigned number of terms being indefinitely small in comparison of n: it is also evident, that they may be rejected in all the rest of the terms of the series; because these terms (by reason of the indefinite smallness of e) bear no assignable proportion to the preceding ones. Hence we have $1 + ne + \frac{n^2e^2}{2} + \frac{n^3e^3}{2.3} + \frac{n^4e^4}{2.3.4}$, &c. $=$ N: but ne is ($=$ L) the hyperbolic logarithm of $\overline{1+e}]^n$ (or N) by what has been already specified: therefore $1 + \mathrm{L} + \frac{\mathrm{L}^2}{2} + \frac{\mathrm{L}^3}{2.3} + \frac{\mathrm{L}^4}{2.3.4} + \frac{\mathrm{L}^5}{2.3.4.5}$, &c. $=$ N. *Q. E. I.*

PROP. II.

To determine the hyperbolic logarithm (L) *of any given number* (N).

It appears from the preceding prop. that $1 + \mathrm{L} + \frac{\mathrm{L}^2}{2} + \frac{\mathrm{L}^3}{2.3}$, &c. is $=$ N: therefore, if $x + 1$ be put $=$

N, we shall have $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4}$, &c. $= x$; and, consequently, by reverting the series, $L = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6}$, &c. *Q. E. I.*

OTHERWISE.

Because $\overline{1+e}]^n = N$ (by the definition of logarithms), we shall have $1 + e = N^{\frac{1}{n}} = \overline{1+x}]^m$; by putting $1 + x =$ N, and $m = \frac{1}{n}$. Therefore, $\overline{1+x}]^m$ being $= 1 + mx + m \cdot \frac{m-1}{2} \cdot x^2 + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot x^3$, &c. we have $e = mx + m \cdot \frac{m-1}{2} \cdot x^2 + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot x^3$, &c. where, m being rejected in the factors $m - 1$, $m - 2$, $m - 3$, &c. as indefinitely small in comparison of 1, 2, 3, &c. the equation will become $e = mx - \frac{mx^2}{2} + \frac{mx^3}{3} - \frac{mx^4}{4}$, &c. whence $\frac{e}{m}$ $(= ne = L) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$, &c. the very same as before.

But this series, though indeed the most easy and natural, is of little use in determining the logarithms of large numbers; since, in all such cases, it diverges, instead of converging. It will be proper, therefore, to give here the invention of other methods, which authors have had recourse to, in order to obtain a series that will always converge.

First, then, let the number whose logarithm you would find be denoted by $\frac{1}{1-x}$; where it is manifest (however great that number may be) x will be always less than unity: moreover, let $\overline{1+e}|^n$ (as before) be the term of the logarithmic progression agreeing with the proposed number, or, which is the same, let $\overline{1+e}|^n = \frac{1}{1-x}$: then, by taking the root on both sides, we shall have $1+e = \frac{1}{\overline{1-x}|^{\frac{1}{n}}} = \overline{1-x}|^{-\frac{1}{n}} = \overline{1-x}|^m$ (by making $m = -\frac{1}{n}$) $= 1 - mx + m \cdot \frac{m-1}{2} \cdot x^2 - m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot x^3$, &c. where m being rejected in the factors $m-1$, $m-2$, &c. (as before), our equation will become $1 + e = 1 - mx - \frac{mx^2}{2} - \frac{mx^3}{3}$, &c. whence $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$, &c. $= \frac{e}{-m} = ne$ = the hyperbolic logarithm of $\frac{1}{1-x}$. Which series, it is manifest, will always converge, let the value of $\frac{1}{1-x}$ be ever so great; because x will be always less than unity.

But it is further observable, that this series has exactly the same form, except in its signs, with that above for the logarithm of $1+x$; and that, if they be added together, the series $2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7}$, &c. thence arising, will be more simple than either of them, since one half of the terms will be entirely destroyed thereby. Therefore, because the sum of the logarithms of any two numbers is equal

to the logarithm of the product of those numbers (*see Article* 1.), it is manifest that $2x + \frac{2x^3}{3} + \frac{2x^5}{5}$, &c. will truly express the logarithm of $\frac{1+x}{1-x}$, or $\frac{1}{1-x} \times \overline{1+x}$. Which series converges still faster than $x + \frac{x^2}{2} + \frac{x^3}{3}$, &c. not only because the even powers are here destroyed, but because x, in finding the logarithm of any given number (N), will have a less value. But now, to determine what this value must be, make $\frac{1+x}{1-x} = \mathrm{N}$, and then x will be found $= \frac{\mathrm{N}-1}{\mathrm{N}+1}$; but if the quantity proposed be a fraction $\left(\frac{\mathrm{P}}{\mathrm{Q}}\right)$, instead of a whole number, make $\frac{\mathrm{P}}{\mathrm{Q}} = \frac{1+x}{1-x}$, and you will have $x = \frac{\mathrm{P}-\mathrm{Q}}{\mathrm{P}+\mathrm{Q}}$: either of which values, substituted in the foregoing series $2x + \frac{2x^3}{3} + \frac{2x^5}{5}$, &c. will give the hyperbolic logarithm of the respective number.

Example. Let it be proposed to find the hyperbolic logarithm of the number 2.

Here, x being $= \frac{2-1}{2+1} = \frac{1}{3}$, and $x^2 = \frac{1}{9}$; we shall have

$$\begin{aligned}
x\ (=\tfrac{1}{3}) &= ,333333333, \text{ \&c.}\\
x^3\ (=\tfrac{1}{9}x) &= ,037037037, \text{ \&c.}\\
x^5\ (=\tfrac{1}{9}x^3) &= ,004115226, \text{ \&c.}\\
x^7\ (=\tfrac{1}{9}x^5) &= ,000457247, \text{ \&c.}
\end{aligned}$$

$$x^9 \ (=\tfrac{1}{9}x^7) \ = ,000050805, \&c.$$
$$x^{11} \ (=\tfrac{1}{9}x^9) \ = ,000005645, \&c.$$
$$x^{13} \ (=\tfrac{1}{9}x^{11}) = ,000000627, \&c.$$
$$x^{15} \ (=\tfrac{1}{9}x^{13}) = ,000000069, \&c.$$
$$\&c. \qquad \&c.$$

Which values being respectively divided by the numbers 1, 3, 5, 7, 9, &c. and the several quotients added together (*see the general series*), we shall have ,346573590, &c. whose double, being ,693147180, &c. is the hyperbolical logarithm of the number 2.

After the very same manner the hyperbolic logarithm of any other number may be determined; but, as the series converges, slower and slower, the higher we go, it is usual, in computing of tables, to derive the logarithms we would find, by help of others already known; for which there are various methods; but the following is the most commodious and simple that has occurred to me, especially when a great degree of accuracy is required.

It is thus. Let a, b, and c denote any three numbers in arithmetical progression, whose common difference is unity; then, a being $= b - 1$ and $c = b + 1$, we shall have $ac = b^2 - 1$, and consequently $\frac{b^2}{ac} = \frac{ac+1}{ac}$. Whence, by the nature of logarithms, we likewise have $2 \log. b - \log. a - \log. c = \log. \frac{ac+1}{ac}$; and putting $\frac{ac+1}{ac} = \frac{1+x}{1-x}$, we have $x = \frac{1}{2ac+1}$; which value being substituted for x in the general series $(2x + \frac{2x^3}{3} + \frac{2x^5}{5}, \&c.)$ the hyp. log. of $\frac{ac+1}{ac}$

will be obtained; which being denoted by S, we shall have

$$\left\{\begin{array}{l} \log.\ b = \frac{1}{2}\log.\ a + \frac{1}{2}\log.\ c + \frac{1}{2}\text{S.} \\ \log.\ a = 2\log.\ b - \quad \log.\ c - \quad \text{S.} \\ \log.\ c = 2\log.\ b - \quad \log.\ a - \quad \text{S.} \end{array}\right\}$$

As an example hereof, let it be proposed to find the hyperbolic logarithm of 3.

Then the hyperbolic logarithm of 2 being already found $= ,693147180$, &c. that of 4, which is the double thereof, will also be known. Therefore, taking $a = 2$, $b = 3$, and $c = 4$, we shall, in this case, have $x\left(\frac{1}{2ac+1}\right) = \frac{1}{17}$, $x^2 = \frac{1}{289}$: whence

$$x\left(=\frac{1}{17}\right) = ,058823529, \text{ \&c.}$$

$$x^3\left(=\frac{x}{289}\right) = ,000203542, \text{ \&c.}$$

$$x^5\left(=\frac{x^3}{289}\right) = ,000000704, \text{ \&c.}$$

&c. &c.

Therefore $\frac{1}{2}\text{S}\left(x + \frac{x^3}{3} + \frac{x^5}{5}, \text{\&c.}\right) = ,058891517$, &c. and consequently hyp. log. 3. $\left(\frac{hyp.\ log.\ 2 + hyp.\ log.\ 4.}{2} + \frac{1}{2}\text{S}\right)$ $= 1,098612288$, &c.

2. Let the hyberbolic logarithm of 10 be required.

The logarithms of 8 and 9 being given, from those of 2 and 3, already found, a may, here, be $= 8$, $b = 9$, and $c = 10$; and then $x\left(\frac{1}{2ac+1}\right)$ being $= \frac{1}{161}$, we shall have $\frac{1}{2}$ S

$(x + \frac{x^3}{3} + \frac{x^5}{5}, \&c.) = ,006211180, \&c. + ,000000079, \&c.$ $\&c. = ,006211259, \&c.$

And therefore log. 10 (2 *log.* 9 — *log.* 8 — S) = 2,302585092, &c.

Hitherto we have had regard to logarithms of the hyperbolic kind: but those of any other kind may be derived from these, by *barely* multiplying by the proper multiplicator, or modulus.

Thus, in the *Brigean* or common form, where a unit is assumed for the logarithm of 10, the logarithm of any number will be found, by multiplying the hyperbolic logarithm of the same number by the fraction ,434294481, &c. which is the proper modulus of this form.

For, since the logarithms of all forms preserve the same proportion with respect to each other, it will be, as 2,302585092, &c. the hyperbolic log. of 10 above found is to (H) the hyperbolic logarithm of any other number, so is 1, the common logarithm of 10, to $\left(\frac{H}{2,302585092, \&c.}\right)$ H × ,434294481, &c. the common logarithm of the same number.

But, to avoid a tedious multiplication, which will always be required when a great degree of accuracy is insisted on, the best way to find the logarithms of this form is from the series $2x + \overline{\frac{2x^3}{3} + \frac{2x^5}{5}, \&c.} \times \overline{0,434294481, \&c.}$ which expresses the common logarithm of $\frac{1+x}{1-x}$ (by what has been

already shown), and which, by making $R = ,868588963$, &c. will stand more commodiously thus, $Rx + \frac{Rx^3}{3} + \frac{Rx^5}{5} + \frac{Rx^7}{7}$, &c.

For an example hereof, let the common logarithm of 7 be required: in which case the logarithms of 8 and 9 being known, from those of 2 and 3, we shall have log. $7 = 2$ log. $8 -$ log. $9 - S$ (*by the Theor.*), S being $= Rx + \frac{Rx^3}{3} + \frac{Rx^5}{5}$, &c. ($=$ the common log. of $\frac{64}{63}$), and $x \left(= \frac{64 - 63}{64 + 63}\right) = \frac{1}{127}$: whence ($x^2$ being $= \frac{1}{16129}$) we shall have

$$Rx \left(= \frac{,8685, \&c.}{127}\right) = ,006839283, \&c.$$

$$Rx^3 \left(= \frac{Rx}{16129}\right) = ,000000424, \&c.$$

$$Rx^5 \left(= \frac{Rx^3}{16129}\right) = ,00000000002, \&c.$$

&c. &c.

Consequently $S \left(Rx + \frac{Rx^3}{3} + \frac{Rx^5}{5}, \&c.\right) = ,006839424$, &c. and 2 log. $8 -$ log. $9 - S = ,845098040$, &c. $=$ the common logarithm of 7 required. But the same conclusion may be brought out by fewer terms of the series, if the logarithms of the three first primes 2, 3, and 5 be supposed known; because those of 48 and 50 (which are composed of them) will likewise be known; from whence the logarithm of 7 $\left(= \frac{1}{2} \text{ log. } 49 = \frac{\textit{log.}\ 48 + \textit{log.}\ 50 + S}{4}\right)$

will come out = ,845098040, &c. (*as before*), which value will be true to 11 places of figures, by taking the first term of the series, *only*.

Again, let the common logarithm of the next prime number, which is 11, be required. Here a may be taken $= 10$, $b = 11$, and $c = 12$; but fewer terms of the series will suffice, if other three numbers, composed of 11 and the inferior primes be taken, whereof the common difference is a unit. Thus, because $98 = 2 \times 7 \times 7$, $99 = 3 \times 3 \times 11$ (9×11), and $100 = 2 \times 2 \times 5 \times 5$ (or 10×10), let there be taken $a = 98$, $b = 99$, and $c = 100$; and then, by the first term of the series only, the log. of 99 will be found true to 14 places; whence that of 11 (log. 99 — log. 9) is also known.

But, notwithstanding all these artifices and compendiums, a method, similar to that in page 21, for finding the logarithms of large numbers, one from another, by addition and subtraction only, still seems wanting in the calculation of tables; I shall, therefore, here subjoin such a method.

1. Let A, B, and C denote any three numbers in arithmetical progression, not less than 10000 each, whereof the common difference is 100.

2. From twice the logarithm of B, subtract the sum of the logarithms of A and C, and let the remainder be divided by 10000.

3. Multiply the quotient by 49,5, and to the product add $\frac{1}{100}$ part of the difference of the logarithms of A and

B; then the sum will be the excess of the logarithm of A + 1 above that of A.

4. From this excess let the quotient, found by Rule 2, be continually subtracted, that is, first from the excess itself, then from the remainder, then from the next remainder, &c.

5. To the logarithm of A add the said excess, and to the sum add the first of the remainders; to the last sum add the next remainder, &c. then the several sums, thus arising, will exhibit the logarithms of A + 1, A + 2, A + 3, &c. respectively.

Thus, let it be proposed to find the logarithms of all the whole numbers between 17900 and 18100; those of the two extremes 17900 and 18100, and that of the mean (18000) being given.

Then the logarithm of $\left\{\begin{matrix} A \\ B \\ C \end{matrix}\right\}$ being equal to $\left\{\begin{matrix} 4{,}252853031 \\ 4{,}255272505 \\ 4{,}257678575 \end{matrix}\right\}$, we shall have $\frac{2 \text{ log. } B - \text{log. } A - \text{log. } C}{10000} = {,}00000000134$ (*see Rule* 2.), which multiplied by 49,5, and the product added to $\frac{\text{log. } B - \text{log. } A}{100}$, gives ,00002426107 for the excess of the logarithm of A + 1 above that of A (*by Rule* 3.) From whence the work, being continued according to Rule 4 and 5, will stand as follows:

,000024	26107	excess.	4,25	2853031	log. 17900
	134			2426107	excess.
	25973	1st. rem.		287729207	log. 17901
	134			2425973	
	25839	2d. rem.		290155180	log. 17902
	134			2425839	
	25705	3d. rem.		292581019	log. 17903
	134			2425705	
	25571	4th. rem.		295006724	log. 17904
	134			2425571	
	25437	5th. rem.		297432295	log. 17905
	134			2425437	
	25303	6th. rem.		299857732	log. 17906
	134			2425303	
	25169	7th. rem.		302283035	log. 17907
	134			2425169	
	25035	8th. rem.		304708204	log. 17908
	134			2425035	
	24901	9th. rem.		307133239	log. 17909
	134			2424901	
	24767	10th. rem.		309558140	log. 17910
	&c.	&c.		&c.	&c.

Note. The logarithms found according to this method, in numbers between 10000 and 20000, are true to 8 or 9 places of figures: those of numbers between 20000 and 50000 err only in the 9th or 10th place; and those of above 50000 are true to 10 places, at least.

Having explained the manner of constructing a table of logarithms, and that by various methods, I [illegible] show the use of such a table in the business of [illegible]

First, in the right-angled plane triangle [illegible] let there be given the hypothenuse AC = 1[illegible] the angle A = 35° 20′; to find the perpendicular [illegible] the base AB.

Here, because radius : sine 35° 20′ : : 17910 : BC (*by Theor.* 1. *p.* 7.), we have BC $= \frac{\textit{sine } 35° \ 20' \times 17910}{\textit{radius}}$; therefore, because the addition and subtraction of logarithms answers to the multiplication and division of the natural numbers (*see p.* 43. 44.), we have *log.* BC = *log. sine* 35° 20′ + *log.* 17910 — *log. radius.*

But, by the tables of artificial or logarithmic sines*, the log. sine of 35° 20′ will appear to be 9,7621775; to which add 4,2530956, the log. of 17910, and from the sum (14,0152731) take 10, the log. of radius, and there results 4,0[illegible]52731 = the log. of BC; which, in the tables, answers to 103[illegible]8, the length of BC required.

Again, for AB, it will be, as radius : sine of C (54° 40′) : : AC (17910) : AB (*by Theorem* 1.) Whence, by adding the logarithms of the second and third terms together, and subtracting that of the first, as above, we have AB = 14611. See the operation.

* A table of artificial [illegible] is nothing more than a table of the logar[illegible]ms of the numbers expressing the natural sines, to the radius 1[illegible] whose logarithm is 10.

Log. radius	-	10,
Log. sine C (54° 40′)		9,9115844
Log. AC (17910)	-	4,2530956
Log. AB = 14611		4,1646800

Moreover, in the oblique plane triangle ABC (fig. 3.), let there be given AB = 75, AC = 60, and the included angle A = 48°; to find the other two angles. Then (*by Theorem* 5.) it will be,

As AB + AC (135)	its log.	2,1303338
is to AB — AC (15)	its log.	1,1760913
so is T. $\frac{C+B}{2}$ (66°)	-	10,3514169
		11,5275082
to the T. $\frac{C-B}{2}$ = 14° 00′	-	9,3971744

Which 14° added to 66°, the half sum of the angles C and B, gives the greater C = 80°; and, subtracted therefrom, leaves the lesser B = 52°.

Lastly, in the right-angled spherical triangle ABC (fig. 17.), let there be given the hypothenuse AC = 60°, and the angle A = 23° 29′; to find the base and perpendicular. Then (*by Theor.* 1. *p.* 28.) the operation will be as follows.

Log. radius - - -	10, [illegible]
Log. sine A (23° 29′) - -	9, [illegible]
Log. sine AC (60°) -	9, [illegible]
Log. sine BC = 20° 11′ -	[illegible]
Also log. radius - -	10 [illegible]
Log. Co-s. A (23° 29′) -	9,9624527 [illegible]
Log. T. AC (60°) -	10,2385606
Log. T. AB = 57° 49′ -	10,2010133

Having exhibited the manner [illegible] the common cases of plane and spherical t[illegible] logarithms and otherwise, I shall here [illegible] positions for the solution o[illegible] more di[illegible] sometimes occur; when [illegible] of the [illegible] themselves, their sums, or differences, [illegible] are [illegible]

[illegible]ROPOSITI[illegible]

[illegible], or versed sine o[illegible] [illegible]en, to find [illegible] sine [illegible] co-sine, &c. of h[illegible]

From the two extremes of th[illegible] [illegible]g. 19.), let the chords AE and BE [illegible] radius CQ bisect AE, perpendicul[illegible]); then will AD be the sine, and [illegible] ACD [illegible] ½ACE.

Bu[illegible] 4AD² = [illegible]
(*by Cor. to* 8. 6[illegible] × AF; [illegible]

AF: also $4CD^2 = BE^2 = AB \times BF = 2AC \times$ [illegible] $CD^2 = \frac{1}{2}AC \times BF$. *From which it appears,* [illegible] *e of the sine of half any arch, or angle, is* [illegible] *angle under half the radius and the versed sine* [illegible] *nd that the square of its co-sine is equal to a* [illegible] *er half the radius and the versed sine of the* [illegible] *the whole arch, or angle.*

PROP. II.

The sines and co-sines of two arches being given, to find the sines and [illegible] *the sum and difference of these arches.*

Let AC an[illegible] 10.) b[illegible] two proposed [illegible]ches; let C[illegible] the sine [illegible]o-sine of the [illegible]r AC, and [illegible] (B*m* and O*m* be those of the [illegible]D (or B[illegible]r, let DG and OG be the sine [illegible]d co-sine[illegible] and BE and OE, those of the difference [illegible] *m* parallel to [illegible]ting AO in *n;* als[illegible] BH parallel to [illegible] in *v* and [illegible]in, because D*m* [illegible] = H*v* [illegible]; and that the triangles O[illegible] O*mn*, and [illegible]ence we have the following proportions [illegible]

COC : O*m* [illegible] × OC = O*m* × CF.
[illegible]C : O[illegible] OC = D*m* × OF.
[illegible] OC = O*m* × OF.
[illegible] = D*m* × C[illegible].

Now, by adding the two first of these equations together, [illegible] + Dv × OC (DG × OC) = Om × CF + Dm [illegible] (DG) is known. Moreover, by taking the [illegible] r, we get mn — Dv × OC (BE × OC) [illegible] whence BE is known.

[illegible] the third and fourth equations [illegible] × OC (OE × OC) = Om × [illegible] ubtracting the latter from the [illegible] OC (OG × OC) = Om × OF [illegible] E and OG are also known.

COROLLARY I.

[illegible] of two arches be denoted by S and s, [illegible] nd radius by R, then will

[illegible] $+ sC$ / R,

[illegible] $= \frac{Sc - sC}{R}$,

[illegible] $= \frac{Cc - Ss}{R}$,

[illegible] difference $= \frac{Cc + Ss}{R}$.

COROLLARY II.

[illegible] sine of the double of either arch (when they are equal) will be $= \frac{2CS}{R}$, and its co-sine $= \frac{C^2 - S^2}{R}$: whence it appears, that *the sine of the double of any arch is equal to twice the rectangle of the sine and co-sine of the single arch, divided by radius; and that its co-sine is equal*

to the difference of the squares of the sine and cosine of the single arch also divided by radius.

COROLLARY III.

Moreover, because $Dm \times$ [illegible] $= \frac{1}{2}OC \times \overline{OE - OG}$); and [illegible] $\times 2On = \frac{1}{2}OC \times \overline{OE + OG}$, [illegible] *of the sines of any two arches* [illegible] *rectangle under half the radius* [illegible] *co-sines, of the sum and differe*[illegible] *the rectangle of their co-sines is equal* [illegible] *under half the radius, and the sum of the* [illegible] *difference of the same arches.*

PROP. [illegible]

The tangents of two arches bei[illegible] *gents of the sum and difference of t*[illegible]

Let AN and AM (fig. 20. and 21.) [illegible] and AB and AC their tangents; also, [illegible] gent of their sum, in the first case, and th[illegible] difference, in the second, and let CF, perpen[illegible] radius DN, be drawn: then, because of th[illegible] triangles BAD and BFC, we shall have

$$\left\{\begin{array}{l} BD \times CF = DA \times BC \\ BD \times BF = BA \times BC \end{array}\right\} \textit{by 4. and 16. 6.}$$

Take each of the last equal quantities from BD^2, and there will remain $BD^2 - BD \times BF\ (BD \times DF) = BD^2 - BA \times BC$: now $BD \times DF\ (BD^2 - BA \times BC) : BD \times CF$

$(DA \times BC) :: DF : CF :: DN (DA) : NE :: DA^2 : DA \times NE$; whence, alternately, $BD^2 - BA \times BC : DA^2 (:: DA \times BC : DA \times NE) :: BC : NE$. But the first term of this proportion, because $BD^2 = DA^2 + BA^2$, will also be expressed by $DA^2 + BA^2 - BA \times BC$, or by $DA^2 + BA^2 - BA \times \overline{AB \pm AC}$; or, lastly, by $DA^2 \mp BA \times AC$: therefore, the three first terms of the proportion being known, the fourth, NE, will likewise be known. *Q. E. I.*

COROLLARY.

Hence, if radius be supposed unity, and the tangents of two arches be denoted by T and t, it follows that the tangent of their sum will be $= \frac{T + t}{1 - tT}$, and the tangent of their difference $= \frac{T - t}{1 + tT}$.

But it will be proper to take notice here, once for all, that, if in these, or any other theorems, the tangent, secant, cosine, co-tangent, &c. of an arch greater than 90 degrees be concerned; then, instead thereof, the tangent, secant, cosine, &c. of an arch, as much below 90 degrees, is to be taken, with a negative sine, according to the observation in page 7.

Thus, for instance, let BA (fig. 22.) be an arch greater than 90°, and let the tangent of the sum of AB and AC be required; supposing T to represent the tangent of AD (the supplement of AB) and t the tangent of AC: then, by writing — T instead of T, in the first of the foregoing theorems,

we shall have tang. of BC $= \frac{-T+t}{1-t\times-T} = \frac{-T+t}{1+tT}$, and therefore tang. DC (— tang. BC) $= \frac{T-t}{1+tT}$; which is the very theorem demonstrated in the 2d case.

PROP. IV.

As the sum of the tangents of any two angles BAC, BAD (fig. 23.) *is to their difference, so is the sine of the sum of those angles to the sine of their difference.*

Let BC and BD be the two proposed tangents, to the radius AB; take Bd = BD, join Ad, and draw DE and dF perpendicular to AC. It is manifest, because Bd = BD, that Ad = AD, and dAB = DAB, and, consequently, that CAd is the difference of the two angles BAC and BAD.

Now, by reason of the similar triangles CDE and CdF, it will be, CD (CB + BD) : Cd (CB — BD) : : DE : dF; but DE and dF are sines of DAE and dAF to the equal radii AD and Ad: *whence the truth of the proposition is manifest.*

COROLLARY.

Hence it also appears, that the base (CD) of a plane triangle is to (Cd) the difference of its two segments, made by letting fall a perpendicular, as the sine of the angle (CAD) at the vertex, to the sine of the difference of the angles at the base.

PROP. V.

In any plane triangle ABC (fig. 24.), *it will be, as the sum of the two sides* plus *the base, is to the sum of the two sides* minus *the base, so is the co-tangent of half either angle at the base to the tangent of half the other angle at the base.*

In AC produced take CD = BC, and let BD be drawn: then (*by Theor. 5. p.* 9.) it will be, AD + AB : AD — AB : : tang. $\frac{ABD + D}{2}$ $\left(\frac{180° - A}{2} = 90° - \frac{1}{2}A\right)$: tang. $\frac{ABD - D}{2}$ $\left(\frac{ABD - CBD}{2} = \frac{ABC}{2}\right)$, that is, AC + BC + AB : AC + BC — AB : : co-tang. $\frac{1}{2}$A : tang. $\frac{1}{2}$ABC. *Q. E. D.*

PROP. VI.

In any plane triangle ABC (fig. 25.), *it will be, as the base* plus *the difference of the two sides is to the base* minus *the same difference, so is the tangent of half the greater angle at the base to the tangent of half the lesser.*

In the lesser side CA, produced, take CD = CB, so that AD may be the difference of the two sides; and let BD be drawn: then it is manifest that the angle CBD will be equal to D: but (*by Theor. 5. p.* 9.) AB + AD : AB — AD : : tangent $\frac{D + DBA}{2} = \frac{CAB}{2}$ (*by* 32. 1.) : tangent $\frac{D - DBA}{2} = \frac{CBD - DBA}{2} = \frac{CBA}{2}$. *Q. E. D.*

PROP. VII.

As the base of any plane triangle ABC (fig. 26.) *is to the sum of the two sides, so is the sine of half the vertical angle to the co-sine of half the difference of the angles at the base.*

In AC, produced, take CD = CB; join B, D, and draw CE parallel to AB, and CF perpendicular to BD.

Since CD = CB, therefore is the angle D = DBC (*by* 5. 1.), and, consequently, half the vertical angle ACB = $\frac{D + CBD}{2}$ (*by* 32. 1.) = D.

Moreover, seeing DCB is = the sum of the angles A and CBA, at the base (*by* 32. 1.), it is evident that BCF (or DCF) is equal to half that sum; and, therefore, as ECF is the excess of the greater ABC = BCE (*by* 29. 1.) above the half sum (BCF), it must, manifestly, be equal to half the difference of the same angles A and CBA.

But (*by Thear.* 3.) AB : AD (AC + BC) : : sine D ($\frac{1}{2}$ACB) : sine ABD = sine CED (*by* 29. 1.) = sine FEC = co-sine ECF. *Q. E. D.*

PROP. VIII.

As the base of any plane triangle ABC (fig. 27.) *is to the difference of the two sides, so is the co-sine of half the vertical angle to the sine of half the difference of the angles at the base.*

In the greater side CA let there be taken CD = CB, and let BD be drawn, and likewise CE, perpendicular to BD. It is manifest, because CD = CB, that CDB and CBD are equal to one another, and that each of them is also equal to half the sum of the angles CBA and A at the base (*by* 32. 1.); therefore ABD, being the excess of the greater CBA above the half sum, must consequently be equal to half the difference of the same angles.

But (*by Theor.* 3.) AB : AD (AC — BC) : : sine B (co-sine DCE, or $\frac{1}{2}$C) : sine ABD. *Q. E. D.*

PROP. IX.

As the difference of the two sides AC, BC (fig. 25.), *of a plane triangle, is to the difference of the segments of the base* AQ, BQ (*made by letting fall a perpendicular from the vertex*), *so is the sign of half the vertical angle to the co-sine of half the difference of the angles at the base.*

For AC — BC : AQ — BQ : : AB : AC + BC (*by L.* 6. *and* 16. 6.) : : sine $\frac{ACB}{2}$: co-sine $\frac{B - A}{2}$ (*by Prop.* 7.) *Q. E. D.*

PROP. X.

As the sum of the two sides of a plane triangle is to the difference of the segments of the base (fig. 25.), *so is the co-sine of half the vertical angle to the sine of half the difference of the angles at the base.*

For AC + BC : AQ — BQ : : AB : AC — BC (*by L.* 6. *and* 16. 6.) : : co-sine of $\frac{ACB}{2}$: sine of $\frac{B - A}{2}$ (*by Prop.* 8.) *Q. E. D.*

PROP. XI.

As the tangent of the vertical angle C *of a plane triangle* ABC (fig. 28.) *is to radius, so is half the base* AB *to a fourth proportional; and as half the base is to the excess of the perpendicular above the said fourth proportional, so is the sine of the vertical angle to the co-sine of the difference of the angles at the base.*

Let ABCE be a circle described about the triangle, and from O, the centre thereof, let OA and OC be drawn; moreover, draw CE parallel to BA, meeting the periphery in E, and DOF perpendicular to AB, meeting EC in F. Then it is evident, that DF will be equal to the perpendicular height of the triangle, AOD equal to the vertical angle ACB, and FOC (= EAC) equal to the difference of the angles (ABC and BAC) at the base.

But (*by Theor.* 2.) as tang. AOD (ACB) : radius : : DA ($\frac{1}{2}$AB) : DO; moreover, as DA : OF (DF — DO) : : sine AOD (ACB) : sine OCF, or co-sine of FOC. *Q. E. D.*

PROP. XII.

As the tangent of the vertical angle of a plane triangle ABC (fig. 28.) *is to radius, so is the base* AB *to a fourth proportional; and as the said fourth proportional is to the sum of the semi-base and the line* CD *bisecting the base, so is the difference of these two to the perpendicular height of the triangle.*

Let a circle be described about the triangle, and from O, the centre thereof, let OA, OC, and OD be drawn; also let CF, parallel to AB, be drawn, meeting DO, produced (if need be) in F. It is evident that DF will be perpendicular to AB, and equal to the height of the triangle. But $DC^2 = OC^2 + OD^2 + 2OD \times OF$ (*by* 12. 2.) $= OA^2 + OD^2 + 2OD \times \overline{DF - OD} = OA^2 - OD^2 + 2OD \times DF = AD^2 + 2OD \times DF$ (*by* 47. 1.); whence, by taking away AD^2 from the first and last of these equal quantities, we have $DC^2 - AD^2 = 2OD \times DF$; or $\overline{DC + AD} \times \overline{DC - AD} = 2OD \times DF$ (*by Cor. to* 5. 2.), and therefore $2OD : DC + AD :: DC - AD : DF$; but (*by Theor.* 2.) as the tang. $AOD = ACB$: radius ($:: AD : OD$) $:: AB : 2OD$. *Q. E. D.*

PROP. XIII.

As twice the rectangle under the base and perpendicular of a plane triangle ABC (fig. 29.) *is to the rectangle under the sum and difference of the base and sum of the two sides, so is radius to the co-tangent of half the vertical angle.*

Let HG, perpendicular to AB, be the diameter of a circle described about the triangle; and let HD and H*d* be perpendicular to the two sides of the triangle; also let CF be parallel to AB, and let HA, HB, and HC be drawn.

Since the diameter HG is perpendicular to AB, therefore is AE = BE (*by* 3. 3.), AH = BH, and the angle ACH = BCH (*by* 28. *and* 27. 3.); whence, also, CD = C*d*, and HD = H*d* (*by* 15. 1.) Therefore the right-angled triangles HAD and HB*d*, having AH = HB and HD = H*d*, have, likewise, AD = B*d* (*by* 26. 1.) From whence it is manifest, that CD will be equal to half the sum, and AD equal to half the difference of the two sides of the triangle. Moreover, because of the similar triangles AEH and HCD, it will be, $AE^2 : CD^2 :: HA^2 = HE \times HG$ (*by Cor. to* 8. 6.) : HC^2 $(HF \times HG) :: HE : HF$ (*by* 1. 6.) Whence, *by division, &c.* $AE^2 : CD^2 - AE^2 :: HE : EF :: HE \times AE : EF \times AE$; therefore, *by inversion and alternation*, $EF \times AE : CD^2 - AE^2$ $\overline{(CD + AE} \times \overline{CD - AE)} :: HE \times AE : AE^2$. HE : AE : : radius : co-tang. EAH (ACH), *by Theor.* 2.: *whence the truth of the proposition is manifest.*

PROP. XIV.

As twice the rectangle of the base and perpendicular of a plane triangle ABC (fig. 29.) *is to the rectangle under the sum and difference of the base, and the difference of the two sides, so is radius to the tangent of half the vertical angle.*

Let the preceding construction be retained, and let AG and CG be drawn. The triangles AHD and GHC (being right-angled at D and C, and having HAD = HGC, *by* 21. 3.) are equiangular; and so AD : GC : : AH : HG : : AE : AG (*by* 8. 6.); whence, alternately, AD : AE : : GC : AG, and $AD^2 : AE^2 : : GC^2$ (GF × HG) : AG^2 (GE × GH) : : GF : GE; therefore, *by division*, &c. $AE^2 - AD^2 : AE^2 : :$ EF : GE : : EF × AE : GE × AE; whence, again, *by alternation*, &c. EF × AE : $AE^2 - AD^2$ $\overline{(AE + AD)} \times \overline{AE - AD)}$: : GE × AE : AE^2 : : GE : AE : : radius : tang. AGE (*by Theor.* 2.); *from which the truth of the proposition is manifest.*

PROP. XV.

If the relation of three right lines a, b, *and* x (fig. 30.) *be such, that* $ax - x^2 = b^2$, *then it will be, as* $\frac{1}{2}a$: b : : *radius to the sine of an angle, and as radius to the tangent* (*or cotangent*) *of half this angle, so is* b : x.

Make AB equal to a, upon which let a semicircle ADB be described; also let CD, equal to b, be perpendicular to AB, and meet the periphery in D (for it cannot exceed the radius of the circle when the proposition is possible): moreover, let AD, BD, and the radius OD be drawn. Because AC × CB = $CD^2 = b^2$ (*by Cor. to* 8. 6.), it is plain that AC × $\overline{a - AC}$, or BC × $\overline{a - BC}$ is also $= b^2$; and, therefore, $x \times \overline{a - x}$ being $= b^2$, it is manifest that x may be equal either to AC or to BC. Now (*by Theor.* 1.) OD ($\frac{1}{2}a$) : CD (b) : : radius : sine DOC, whose half is equal to A or BDC (*by* 20. 3.) But as radius : tang. BDC : : DC

(b) : BC; or, as radius : co-tang. BDC (tang. CDA) : : DC (b) : AC. *Q. E. D.*

PROP. XVI.

If the relation of three lines a, b, *and* x (fig. 31.) *be such, that* $x^2 \pm ax = b^2$, *then it will be, as* $\frac{1}{2}a : b :: $ *radius to the tangent of an angle, and as radius is to the tangent or co-tangent of half this angle* (*according as the sign of* ax *is positive or negative*) : : b : x.

Make AB $= b$, and AC perpendicular to AB, equal to a; about the latter of which, as a diameter, let a circle be described; and through O, the centre thereof, let BD be drawn, meeting the periphery in E and D; also let A, E and C, E be joined, and draw BF parallel to AC, meeting AE, produced, in F. Then, since (*by* 36. 3.) BE $\times$ BD $(= \text{BE} \times \overline{\text{BE} + a} = \text{BD} \times \overline{\text{BD} - a}) = \text{AB}^2$ (b^2) and $x \times \overline{x \pm a} = b^2$, by supposition, it is manifest, that BE will be $= x$, when $x \times \overline{x + a} = b^2$; and BD $= x$, when $x \times \overline{x - a} = b^2$.

Furthermore, because the angle F = OAE (*by* 29. 1.) = OEA (*by* 5. 1.) = BEF (*by* 15. 1.), it is evident that BF = BE (*by* 6. 1.), and that the angles BAF and C (being the complements of the equal angles F and OAE) are likewise equal.

Now (*by Theor.* 2.) AO ($\frac{1}{2}a$) : AB (b) : : radius : tang. AOB; whose half is equal to C or BAF. But, as radius : tang. BAF : : AB (b) : BF (BE), the value of x in the

first case, where $x^2 + ax = b^2$. Again, radius : co-tang. BAF (tang. F) : : BF (BE) : AB (*by Theor.* 2.); and BE : AB : : AB : BD (*by Cor. to* 36. 3. *and* 16. 6.); whence, by equality, radius : co-tang. BAF : : AB (b) : BD; which is the value of x in the second case; where $x^2 - ax = b^2$. *Q. E. D.*

PROP. XVII.

In any plane triangle ABC (fig. 32.), *it will be, as the line* CE, *bisecting the vertical angle, is to the base* AB, *so is the secant of half the vertical angle* ACB *to the tangent of an angle; and, as the tangent of half this angle is to radius, so is the sign of half the vertical angle to the sign of either angle, which the bisecting line makes with the base.*

Let ACBD be a circle described about the triangle, and let CE be produced to meet the periphery thereof in D; moreover, let AD and BD be drawn, and likewise DF, perpendicular to the base AB; which will also bisect it, because (BCD being = ACD) the subtenses BD and AD are equal (*by* 26. *and* 29. 3.) Moreover, since the angle DBE = ACD (*by* 21. 3.) = DCB, the triangles DEB and DBC (having D common) are equiangular, and therefore $DE \times DC = DB^2$ (*by* 4. *and* 17. 6.), or, which is the same, $DE^2 + DE \times CE = DB^2$. Therefore (*by Prop.* 16.) $\frac{1}{2}$CE : DB : : radius : tangent of an angle (which we will call Q); and as radius : tang. $\frac{1}{2}$Q : : DB : DE.

But DB : BF ($\frac{1}{2}$AB) : : secant FBD (BCE) : radius; therefore, by compounding this with the first proportion,

we have $\frac{1}{2}$CE × DB : $\frac{1}{2}$AB × DB : : radius × secant BCE : radius × tang. Q (*by* 22. 6.), and consequently CE : AB : : secant BCE : tang. Q (*by* 1. 6.) Again, BE : DB : : sine DBE (BCE) : sine of DEB (or of CEB); whence, by equality, tang. $\frac{1}{2}$Q : radius : : sine BCE : sine CEB. Q. *E. D.*

PROP. XVIII.

In any plane triangle ABC (fig. 33.), *it will be, as the perpendicular is to the sum of the two sides, so is the tangent of half the angle at the vertex to the tangent of an angle; and as radius is to the tangent of half this angle, so is the sum of the two sides to the base of the triangle.*

Let DP, perpendicular to AB, be the diameter of a circle described about the triangle; let CF be perpendicular to DP, and DG to AC, and let DA, DC, and DB be drawn, and also FI, parallel to BD, meeting BA, produced, in I.

It is manifest, *from Prop.* 13. that CG is equal to half the sum of the sides AC and BC: it also appears (*from* 29. 1. *and* 27. 3.) that the triangles DCG, ADE, BDE, and IFE are all equiangular. Therefore it will be, $CG^2 : BE^2 : : DC^2 (DF \times DP) : BD^2 (DE \times DP) : : DF (DE + EF) : DE : : BE + EI : EB : : \overline{BE + EI} \times BE : EB^2$; and, consequently, $CG^2 (= \overline{BE + EI} \times BE) = BE^2 + EI \times BE$. But, *by Prop.* 16. it will be, $\frac{1}{2}$EI : CG : : radius : tangent of an angle (which we will call Q); and as radius : tang. $\frac{1}{2}$Q : : CG : BE (: : 2CG, or AC + BC, to AB).

But (*by Theor.* 2.) EF : EI : : tang. I (ACD) : radius; therefore, by compounding this proportion with the last but one, we shall have $\frac{1}{2}$EI × EF : EI × CG : : tang. ACD × radius : tang. Q × radius (*by* 22. 6.), and consequently EF : 2CG (AC + BC) : : tang. ACD : tang. Q. *Whence the truth of the proposition is manifest.*

PROP. XIX.

In any plane triangle ABC (fig. 34.), *it will be, as the perpendicular is to the difference of the two sides, so is the co-tangent of half the vertical angle to the tangent of an angle; and, as radius is to the co-tangent of half this angle, so is the difference of the sides to the base of the triangle.*

Let DP, DG, CF, &c. be as in the preceding proposition; also, let PA and PC be drawn, and FI, parallel to PA, meeting AB in I.

The right-angled triangles ADG and DPC, having DAG = DPC (*by* 21. 3.) are similar; and therefore AG : AD : : PC : DP (*by* 4. 6.), and, alternately, AG : PC : : AD : DP : : AE : AP (*by* 8. 6.); whence (*by* 16. 5. *and* 22. 6.) $AG^2 : AE^2$: : PC^2 (PF × PD) : AP^2 (PE × PD) : : PF : PE : : AI : AE : : AI × AE : AE^2 (*by* 1. 6.); and, consequently, AG^2 = AI × AE = AE^2 — EI × AE. Therefore, *by Prop.* 16. $\frac{1}{2}$EI : AG : : radius : tangent of an angle (Q); and as radius : co-tang. $\frac{1}{2}$Q : : AG : AE. But (*by Theor.* 2.) EF : EI : : co-tang. EFI (ACD) : radius; which proportion being compounded with the last but one,

&c. we shall have, EF : 2AG (AC — BC, *see Prop.* 13.) : : co-tang. EFI : tang. Q; and as radius : co-tang. $\frac{1}{2}$Q : : AG : AE : : 2AG (AC — BC) : 2AE (AB). *Q. E. D.*

PROP. XX.

The hypothenuse AC, *and the sum or difference of the legs* AB, BC, *of a right-angled spherical triangle* ABC (fig. 35.), *being given, to determine the triangle.*

Let AE be the sum, and AF the difference of the two legs. Because radius : co-s. AB : : co-s. BC : co-s. AC (*by Theor.* 2.); therefore co-s. AB × co-s. BC = rad. × co-s. AC; but the former of these is = $\frac{1}{2}$ rad. × $\overline{\text{co-s. AE + co-s. AF}}$ (*by Cor.* 3. *to Prop.* 2.); therefore 2 × co-s. AC = co-s. AE + co-s. AF. Whence it appears, *that, if from twice the co-sine of the hypothenuse the co-sine of the given sum or difference of the legs be subtracted, the remainder will be the co-sine of an arch, which, added to the said sum or difference, gives the double of the greater leg required.*

COROLLARY.

Hence, if the two legs be supposed equal to each other (or the given difference = 0), then will the co-sine of the double of each be equal to twice the co-sine of the hypothenuse *minus* the radius.

PROP. XXI.

One leg, BC (fig. 35.), *and the sum or difference of the hypothenuse, and the other leg*, AB, *being given, to determine the hypothenuse.*

Since rad. : co-sine BC : : co-sine AB : co-sine AC (*by Theor.* 2.), it will be (*by comp. and div.*) radius + co-sine BC : rad. — co-s. BC : : co-s. AB + co-s. AC : co-s. AB — co-s. AC. But the radius may be considered as the sine of an arch of 90°, or the co sine of 0; and, therefore, since (*by the lemma in p.* 32.) co-sine 0 + co-sine BC : co-s. 0 — co-s. BC : : co-tang. $\frac{BC + 0}{2}$: tang. $\frac{BC - 0}{2}$; and co-sine AB + co-s. AC : co-s. AB — co-s. AC : : co-tang. $\frac{AC + AB}{2}$: tang. $\frac{AC - AB}{2}$; it follows, *by equality*, that co-tang. $\frac{BC}{2}$: tang. $\frac{BC}{2}$: : co-tang. $\frac{AC + AB}{2}$: tang. $\frac{AC - AB}{2}$; that is, *As the co-tang. of half the given leg is to its tangent, so is the co-tangent of half the sum of the hypothenuse, and the other leg, to the tangent of half their difference.*

PROP. XXII.

The angle at the base, and the sum or difference of the hypothenuse and base of a right-angled spherical triangle (fig. 17.) *being given, to determine the triangle.*

First, it will be, radius : co-sine A : : tang. AC : tang. AB (*by Theor.* 1.), and therefore radius + co-sine A : radius — co-sine A : : tang. AC + tang. AB : tang. AC — tang. AB; whence, by arguing as in the last Prop. it will appear, that co-tang. $\frac{1}{2}$A : tang. $\frac{1}{2}$A : : radius + co-sine A : radius — co-sine A (: : tang. AC + tang. AB : tang. AC — tang. AB) : : sine $\overline{AC + AB}$: sine $\overline{AC - AB}$ (*by Prop.* 4). Hence it appears, that, *As the co-tangent of half the given angle is to its tangent, so is the sine of the sum of the hypothenuse and adjacent leg to the sine of their difference.*

PROP. XXIII.

The hypothenuse AC (fig. 36.) *and the sum or difference of the two adjacent angles being given, to find the angles.*

Let EC be perpendicular to BC; then it will be, radius : co-sine AC : : tang. A : tang. ACE (*by Theor.* 5.) From whence, by reasoning as above, we shall also have co-tang. $\frac{1}{2}$AC : tang. $\frac{1}{2}$AC : : sine $\overline{A + ACE}$: sine $\overline{A - ACE}$; whereof the two last terms, by substituting 90° — ACB for ACE, will become sine $\overline{90° + A - ACB}$ (co-sine $\overline{ACB - A}$) and sine A + ACB — 90° respectively. Whence it appears, that, *As the co-tangent of half the hypothenuse is to its tangent, so is the co-sine of the difference of the angles at the hypothenuse to the sine of the excess of their sum above a right angle.*

COROLLARY.

Hence, if the angles be supposed equal, then it will be, as radius : tangent $\frac{1}{2}$AC : : tangent $\frac{1}{2}$AC : sine $\overline{2A - 90^\circ}$.

PROP. XXIV.

In two right-angled spherical triangles ABC, ADE (fig. 37.), *having one angle* A *common, let there be given the two perpendiculars* BC, DE, *and the sum or difference of the hypothenuses* AC, AE, *to determine the triangles.*

It is evident (*from Theor.* 1.) that sine DE : sine BC : : sine AE : sine AC; therefore sine DE + sine BC : sine DE — sine BC : : sine AE + sine AC : sine AE — sine AC; whence (*by the lemma in p.* 32. *and equality*) tang. $\frac{DE + BC}{2}$: tang. $\frac{DE - BC}{2}$: : tang. $\frac{AE + AC}{2}$: tang. $\frac{AE - AC}{2}$: that is, *As the tangent of half the sum of the two perpendiculars is to the tangent of half their difference, so is the tangent of half the sum of the two hypothenuses to the tangent of half their difference.*

PROP. XXV.

In two right-angled spherical triangles ABC, ADE (fig. 37.), *having the same angle* A *at the base, let there be given the two perpendiculars* BC, DE, *and the sum or difference of the bases* AB, AD, *to determine the bases.*

Since tang. DE : tang. BC : : sine AD : sine AB (*by Theor.* 4. *and equality*); therefore is tang. DE + tang. BC : tang. DE — tang. BC : : sine AD + sine AB : sine AD — sine AB; whence (*by Prop.* 4. *and the lemma in p.* 32.) it will be, sine $\overline{DE + BC}$: e sin $\overline{DE - BC}$: : tang. $\frac{AD + AB}{2}$: tang. $\frac{AD - AB}{2}$; that is,

As the sine of the sum of the two perpendiculars is to the sine of their difference, so is the tangent of half the sum of the two bases to the tangent of half their difference.

PROP. XXVI.

The product of the square of radius and the co-sine of the base of any spherical triangle ABC (fig. 38.) *is equal to the product of the sines of the two sides and the co-sine of the vertical angle, together with the product of radius and the co-sines of the same sides.*

For, let AD be perpendicular to BC; then, since co-s. BD $= \frac{\text{S. CB} \times \text{S. CD} + \text{co-s. CB} \times \text{co-s. CD}}{\text{rad.}}$ (*by Cor.* 1. *to Prop.* 2.), it is evident, that co-s. BD : co-s. CD : : $\frac{\text{S. CB} \times \text{S. CD} + \text{co-s. CB} \times \text{co-s. CD}}{\text{rad.}}$: co-s. CD : : $\frac{\text{S. CB} \times \text{S. CD}}{\text{co-s. CD}}$ + co-s. CB : rad. (*by mult. each term by* $\frac{\textit{rad.}}{\textit{co-s. CD}}$). But $\frac{\text{S. CD}}{\text{co-s. CD}} = \frac{\text{T. CD}}{\text{rad.}}$, therefore our propor-

tion will be $\frac{\text{S. CB} \times \text{T. CD}}{\text{rad.}} + \text{co-s. CB} : \text{rad.}$ (: : co-s. BD : co-s. CD) : : co-s. AB : co-s. AC (*by Cor. to Theor.* 2.); whence, by multiplying means and extremes, we have co-s. $\text{AB} \times \text{radius} = \frac{\text{S. CB} \times \text{co-s. AC} \times \text{T. CD}}{\text{rad.}} + \text{co-s. AC} \times$ co-s. BC. But (*by Theor.* 1.) radius : co-s. C : : T. AC : T. $\text{CD} = \frac{\text{co-s. C} \times \text{T. AC}}{\text{rad.}} = \frac{\text{co-s. C} \times \text{S. AC}}{\text{co-s. AC}}$ (*by Cor.* 1. *to Prop.* 1. *p*, 14.), which last being substituted for its equal, we shall have, co-s. $\text{AB} \times \text{rad.} = \frac{\text{S. CA} \times \text{S. CB} \times \text{co-s. C}}{\text{rad.}}$ $+ \text{co-s. AC} \times \text{co-s. BC}$; from whence, if each term be multiplied by radius, the truth of the proposition will appear manifest.

There is another way of demonstrating this proposition, from the orthographic projection of the sphere; but that is a subject which neither room nor inclination will permit me to treat of here.

PROP. XXVII.

If AE *be the sum, and* AF *the difference, of the two sides of a spherical triangle* ABC (fig. 39.), *and* V *be put to denote the versed sine of the vertical angle, and* R *the radius; then will* $\text{V} = \frac{\text{R}^2 \times \overline{\textit{co-s.}\ \text{AF} - \textit{co-s.}\ \text{AB}}}{\textit{S.}\ \text{AC} \times \textit{S.}\ \text{BC}}$

$$= \frac{2\text{R} \times \overline{\textit{co-s.}\ \text{AF} - \textit{co-s.}\ \text{AB}}}{\textit{co-s.}\ \text{AF} - \textit{co-s.}\ \text{AE}}$$

$$= \frac{2\text{R} \times \textit{S.}\ \overline{\tfrac{1}{2}\text{AB} + \tfrac{1}{2}\text{AF}} \times \textit{S.}\ \overline{\tfrac{1}{2}\text{AB} - \tfrac{1}{2}\text{AF}}}{\textit{S.}\ \text{AC} \times \textit{S.}\ \text{BC}}.$$

It appears from the last Prop. (fig. 40.) that co-s. AB$\times$R is $=$ co-s. AC $\times$ co-s. BC $+ \frac{\text{S. AC} \times \text{S. BC} \times \text{co-s. C}}{\text{R}}$; in which, for co-s. C let its equal R — V be substituted, and then we shall have co-sine AB $\times$ R $=$ co-sine AC $\times$ co-sine BC $+$ sine AC $\times$ sine BC $- \frac{\text{S. AC} \times \text{S. BC} \times \text{V}}{\text{R}}$: but the sum of the two former of the three last terms is $=$ co-s. AF $\times$ R (*by Cor.* 1. *to Prop.* 2.); therefore it will be co-s. AB $\times$ R $=$ co-s. AF $\times$ R $- \frac{\text{S. AC} \times \text{S. BC} \times \text{V}}{\text{R}}$, and consequently $\text{V} = \frac{\text{R}^2 \times \overline{\text{co-s. AF} - \text{co-s. AB}}}{\text{S. AC} \times \text{S. BC}}$. *Which is the first case.* Again, because S. AC $\times$ S. BC is $= \frac{1}{2}$R $\times \overline{\text{co-s. AF} - \text{co-s. AE}}$ (*by Cor.* 3. *to Prop.* 2.), we shall also have $\text{V} = \frac{2\text{R} \times \overline{\text{co-s. AF} - \text{co-s. AB}}}{\overline{\text{co-s. AF} - \text{co-s. AE}}}$. *Which is the second case.* Moreover, since $\frac{1}{2}\text{R} \times \overline{\text{co-s. AF} - \text{co-s. AB}}$ is $= \text{S.} \frac{\text{AB} + \text{AF}}{2} \times \text{S.} \frac{\text{AB} - \text{AF}}{2}$ (*by the same*), it follows that V is likewise $= \frac{2\text{R} \times \text{S.} \overline{\frac{1}{2}\text{AB} + \frac{1}{2}\text{AF}} \times \text{S.} \overline{\frac{1}{2}\text{AB} - \frac{1}{2}\text{AF}}}{\text{S. AC} \times \text{S. BC}}$.

Q. E. D.

COROLLARY I.

Hence, because $\frac{1}{2}\text{R} \times \text{V}$ is $=$ the square of the sine of $\frac{1}{2}$C (*by Prop.* 1.), it follows that sq. S. $\frac{1}{2}\text{C} =$

$$\frac{\text{R}^2 \times \text{S.} \overline{\frac{1}{2}\text{AB} + \frac{1}{2}\text{AF}} \times \text{S.} \overline{\frac{1}{2}\text{AB} - \frac{1}{2}\text{AF}}}{\text{S. AC} \times \text{S. BC}}.$$

From whence we have the following theorem, for solving the 11th case of oblique triangles, where the three sides are given, to find an angle.

As the rectangle of the sines of the two sides, including the proposed angle, is to the rectangle under the sines of half the base plus *half the difference of the sides, and half the base* minus *half the difference of the sides, so is the square of radius to the square of the sine of half the required angle.*

COROLLARY II.

Moreover, because $V = \frac{R^2 \times \overline{\text{co-s. } AF - \text{co-s. } AB}}{S.\ AC \times S.\ BC}$, we shall have $R^2 : S.\ AC \times S.\ BC :: V : \text{co-s. } AF - \text{co-s. } AB$: which gives the following theorem, for finding a side when the opposite angle and the other two sides are given.

As the square of radius is to the rectangle of the sines of the two sides including the given angle, so is the versed sine of that angle to the difference of the co-sines (or versed sines) of the difference of those sides, and the side required.

COROLLARY III.

Again, because $V = \frac{2R \times \overline{\text{co-s. } AF - \text{co-s. } AB}}{\text{co-s. } AF - \text{co-s. } AE}$, we shall, by transforming the equation, and putting W for $(2R - V)$ the versed sine of BCE (the supplement of the vertical angle), have co-sine $AE =$
$\frac{2R \times \text{co-s. } AB - W \times \text{co-s. } AF}{V}$, and the co-s. $AF =$

$\frac{2R \times \text{co-s. } AB - V \times \text{co-s. } AE}{W}$. From whence the sides themselves may be determined, when their sum or difference is given, with the base and vertical angle.

COROLLARY IV.

Lastly, since $2R - W(V) = \frac{2R \times \overline{\text{co-s. } AF - \text{co-s. } AB}}{\text{co-s. } AF - \text{co-s. } AE}$, we have $W \times \overline{\text{co-s. } AF - \text{co-s. } AE} = 2R \times \overline{\text{co-s. } AB - \text{co-s. } AE}$; whence, by multiplying both sides by $\frac{R^2}{4}$, $W \times \frac{1}{2}R \times \frac{1}{2}R \times \overline{\text{co-s. } AF - \text{co-s. } AE}$ ($= \text{co-s.}^2 \frac{1}{2}ACB \times S.\ AC \times S.\ BC$, *by Prop.* 1. *and Cor.* 3. *Prop.* 2.) $= R^2 \times \frac{1}{2}R \times \overline{\text{co-s. } AB - \text{co-s. } AE} = R^2 \times S.\ \frac{AB + AE}{2} \times S.\ \frac{AE - AB}{2}$ (*by Prop.* 3.) $= R^2 \times S.\ \frac{AB + AC + BC}{2} \times S.\left(\frac{AB + AC + BC}{2} - AB\right)$. Whence the following analogy; *As the rectangle of the sines of the two sides is to the square of the radius, so is the rectangle of the sines of half the sum of the three sides, and of the excess of that half sum above the base, to the square of the co-sine of half the vertical angle.*

APPENDIX.

SPHERICAL PROJECTIONS.

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DEFINITIONS.

1. The projection of the sphere is the delineation of the circles drawn upon its surface, as seen upon a given plane, called the plane of projection, the position of the eye being also given. There are several kinds of projection, according to the situation of the eye, and the plane to which the circles are referred.

2. In the orthographic projection, the plane of projection passes through the centre of the sphere; and the eye is supposed to be so situated, that all lines drawn to it from any points on the sphere, are perpendicular to the plane of projection.

3. In the stereographic projection, the plane of projection passes through the centre of the sphere, and the eye is situated in the pole of the circle formed by the intersection of this plane and the spherical surface.

4. The primitive circle is that on whose plane the projections are made; or it is the common section of the plane

of projection and the surface of the sphere. And the pole of this circle is called the pole of projection. The point in which the eye is supposed to be situated is called the projecting point.

5 The line of measures of any circle, is the common section of the plane of projection and another plane which passes through the axes of the primitive and of that circle.

6. The right line drawn from any point on the sphere to the projecting point, is called a projecting line.

7. A circle of the sphere is called an original circle; and its representative on the plane of projection, a projected circle.

SECTION I.

The Orthographic Projection of the Sphere.

PROP. I.

If a right line AB (fig. 40.) *be projected upon a plane, it will be projected into a right line; and its length will be to the length of the projection, as radius to the co-sine of its inclination to that plane.*

For let fall the perpendiculars A*a*, B*b* upon the plane of projection; then *ab* will be the line into which it is projected; but, by trigonometry, AB : A*o*, or *ab* : : radius : sine of B, or co-sine of *a*AB.

COROLLARY I.

A right line projected upon a plane parallel thereto, is projected into a right line parallel and equal to itself.

COROLLARY II.

An angle projected upon a plane, which is parallel to the lines forming the angle, is projected into an angle equal to itself.

COROLLARY III.

Any plane figure projected upon a plane parallel to itself, is projected into a figure similar and equal to itself.

COROLLARY IV.

Hence also the area of any plane figure, is to the area of its projection : : radius : co-sine of its inclination to the plane of projection.

PROP. II.

A circle perpendicular to the plane of projection, is projected into a right line equal to its diameter.

For projecting lines drawn through all the points of the circle fall in the common section of the planes of the circle and of projection, which is a right line (3. 2. Supp.), and

that line is equal to the diameter of the circle; because the planes intersect in that diameter. *Q. E. D.*

COROLLARY.

Hence any plane figure perpendicular to the plane of projection is projected into a right line. For the perpendiculars from every point, will all fall in the common section of the plane of the figure and the plane of projection.

PROP. III.

A circle parallel to the plane of projection is projected into a circle equal to itself, and concentric with the primitive.

The circle being parallel to the primitive, their common axis passes through the projecting point (2. and 4. def.), and every radius of the circle is projected into a line equal to itself (Cor. 1. prop. 1.) *Q. E. D.*

COROLLARY.

The radius of the projected circle is the co-sine of the distance of its original circle from the primitive, and the sine of its distance from its own pole.

PROP. IV.

An inclined circle is projected into an ellipsis whose transverse axis is the diameter of the circle.

Let ADBH (fig. 41.) be the inclined circle, P its centre; and let it be projected into *adbh*; draw the plane ABFC*a* through the centre C of the sphere, perpendicular to the plane of the given circle and plane of projection, to intersect them in the lines AB, *ab*; draw GPH, DE perpendicular, and DQ parallel to AB; then, because the line GP, and the plane of projection, are both perpendicular to the plane ABF, GH is parallel to the plane of projection, and therefore to *gh*.

In the circle ADB, $DQ^2 = GQH = gqh$, and $BP^2 = GP^2 = gp^2$. And (16. 2. *Supp.*) BP : EP or DQ : : *bp* : *ep* or *dq*, and $BP^2 : DQ^2 :: bp^2 : dq^2$; that is, $gp^2 : gqh :: bp^2 : dq^2$; and therefore *agbh* is an ellipsis, whose transverse *gh* is the diameter of the circle (*Cor.* 2. *to* 15. 2. *Conics*). *Q. E. D.*

COROLLARY I.

Since *ab* is perpendicular to *gh*, therefore *ab* is the conjugate axis, and is twice the sine of the angle AB*b* to the radius *gp*: that is, the conjugate axis is equal to twice the co-sine of the inclination, to the radius of the circle.

COROLLARY II.

The transverse axis is equal to twice the co-sine of its distance from its parallel great circle. For $gh = GH = 2\,AP =$ twice the sine of AK.

COROLLARY III.

The extremities of the conjugate axis are distant from the centre of the primitive, by the sines of the circles nearest and greatest distance from the pole of the primitive. Thus *a*C is the sine of AN, and *b*C the sine of BN.

COROLLARY IV.

Hence also it is plain, that the conjugate axis always passes through the centre C of the primitive; and is always in the line of measures of that circle.

SCHOLIUM.

Every circle in the projection represents two equal circles, parallel and equidistant from the primitive. Every right line represents two semicircles, one towards the eye, the other in the opposite side. Every ellipsis represents two equal circles, but contrarily inclined, as AB, CD; one above the primitive and the other below it.

SECTION II.

The Stereographic Projection of the Sphere.

PROP. I.

Any circle passing through the projecting point, is projected into a right line.

For lines drawn from the projecting point to any part of the circle will be in its plane; and will therefore meet the plane of projection in the common section of that plane and the plane of the circle, which is a right line (3. 2 Supp.)

COROLLARY.

A great circle passing through the poles of the primitive is projected into a right line passing through the centre.

PROP. II.

Any point on the sphere is projected into a point, distant from the centre of the primitive the semi-tangent of its distance from the pole opposite to the projecting point.

Let E (fig. 42.) be the point to be projected; A the projecting point; M the opposite pole; BH the plane of projection (seen edgewise); ABEM a great circle perpendicular to the plane of projection. Since AM is the axis of the primitive, AC is at right angles to BH; therefore GC is the tangent of GM, or the semi-tangent of EM.

COROLLARY I.

Any circle passing through the projecting point is projected into a right line perpendicular to the line of measures, and distant from the centre the semi-tangent of its nearest distance from the pole opposite to the projecting point.

Thus if AE be a circle passing through A, and at right angles to ABEM, the common section of AE and the plane of projection will be perpendicular to the line of measures (def. 5. and 18. 2 Supp.), and its distance GC is the semi-tangent of EM.

COROLLARY II.

Any arc EM of a great circle perpendicular to the primitive, is projected into the semi-tangent of it. Thus EM is projected into GC.

COROLLARY III.

A great circle perpendicular to the primitive, is projected into a line of semi-tangents passing through the centre, and produced infinitely.

For MF is projected into its semi-tangent CH, and EM into the semi-tangent CG.

COROLLARY IV.

Any arc EMF of a great circle perpendicular to the primitive, is projected into the sum or difference of the

semi-tangents of its greatest and least distances from the pole opposite the projecting point, according as the extremities lie on different or the same side of that pole.

PROP. III.

Every circle, that passes not through the projecting point, is projected into a circle.

Let the original circle EF (fig. 43.) be parallel to the primitive BD; lines drawn from all points of it to the projecting point A, will form a conical surface, which being cut by the plane BD parallel to the base, the section GH (into which EF is projected) will be a circle. (Simson's Conics, book 1. prop. 23.)

Case 2. Let ABMF (fig. 42.) be a great circle perpendicular to the primitive and to the circle EF, A the projecting point, BH the line of measures to the circle EF; draw FK parallel to BD, then the arc AK=AF, and therefore angle AFK or AHG = AEF; therefore in the triangles AEF, AGH, the angles at E and H are equal, and the angle A common, therefore the angles at F and G are equal. Therefore the cone of rays AEF (whose base EF is a circle) is cut by sub-contrary section, by the plane of projection BD, and therefore the section GH (which is the projection of the circle EF, will also be a circle. (*Simson's Conics, book 1, prop.* 24.) *Q. E. D.*

COROLLARY I.

When AF is equal to AG, the circle EF is projected into a circle equal to itself.

For the *similar* triangles AHG and AEF, will also be *equal*, and GH = EF.

COROLLARY II.

Since the plane AB, MF is perpendicular to EF, the former passes the pole of the latter, and therefore the projected pole of the original circle, and the centre of its representative, are in the line of measures.

PROP. IV.

The angle made by two circles on the surface of the sphere, is equal to that made by their representatives upon the plane of projection.

Let the angle BPK (fig. 44.) be projected. Through the angular point P and the centre C, draw the plane of a great circle PED, perpendicular to the plane of projection EFG; let a plane PHG touch the sphere in P; then since the circle EPD is perpendicular both to this plane and to the plane of projection, it is perpendicular to their common section GH. The angles made by circles are the same as those made by their tangents; therefore in the plane PGH, draw the tangents PH, PF, PG to the arcs PB, PD, PK, and these will be projected into the lines pH, pF, pG. Now I say the angle HPG = angle HpG; for the angle CPF = a right angle = CpA + CAp; therefore, taking away the equal angles CPA and CAP, angle pPF=CpA or PpF; consequently pF=PF. Therefore, in the right-angled triangles PFG and pFG, there are two sides equal and the inclu-

ded right angle; whence the hypothenuse PG = *p*G. And for the same reason, in the right-angled triangles PFH and *p*FH, PH=*p*H. Lastly, in the triangles PHG and *p*HG, all the sides are respectively equal, and therefore angle P = angle *p*. *Q. E. D.*

COROLLARY.

The angle made by two circles on the sphere, is equal to the angle made by the radii of their projections at the point of intersection.

For the angle made by two circles on a plane, is the same with that made by their radii drawn to the point of intersection.

PROP. V.

The centre of a projected (lesser) circle, perpendicular to the primitive, is in the line of measures, distant from the centre of the primitive, the secant of the lesser circle's distance from its own pole; and its radius is the tangent of that distance.

Let ABLK (fig. 45.) be the primitive circle, EF the common section of the plane of this circle and that of the less circle, CH the line of measures. The centre of the projected circle is in CH (*Cor.* 2. *Prop.* 3.) Since E and F are in the primitive, the projected circle must evidently pass through E and F. Let D be the centre of the projected circle, and join CF, DF; then CFD is a right angle (*Cor.*

Prop. 4.); therefore FD is the tangent, and CD the secant of FB. *Q. E. D.*

PROP. VI.

The centre of projection of a great circle is in the line of measures, distant from the centre of the primitive, the tangent of its inclination to the primitive; and its radius is the secant of its inclination.

Let AIB (fig. 46.) be the primitive, C its centre, AGB the projected great circle, IH its line of measures. The centre is in IH (*Cor.* 2. *prop.* 3.) Let D be the centre; join CA, DA; then the angle CAD is equal to the inclination of the circles (*Cor. to prop.* 4.) But ACD is a right angle (*Def.* 5. *and* 18. 2 *Supp*); therefore CD is the tangent, and AD the secant, of the inclination of the circle AGB to the primitive. *Q. E. D.*

COROLLARY.

Of all projected great circles, the primitive is the least. For the radius of any oblique great circle being the secant of its inclination to the primitive, is greater than the radius of the primitive.

PROP. VII.

If a projected great circle pass through G (fig. 47.), *one extremity of the diameter of a projected great circle* GAH,

its centre will be in the right line RS, *drawn through* D, *the centre of* GAH, *perpendicular to* GH.

For since all great circles cut each other at a semicircle's distance, all great circles passing through G must cut at the opposite point H; and therefore their centres must be in the line RDS. *Q. E. D.*

COROLLARY I.

Hence, also, if any oblique circle GLH be required to make any given angle with another circle BGAH, it will be projected the same way with regard to GAH considered as a primitive, and RS its line of measures; as the circle BGA is on the primitive BIA, and line of measures ID. And therefore the tangent of the angle AGL to the radius GD, set from D to N, gives the centre of GL.

For the angle NGD will then be equal to AGL (*by Cor. prop.* 4.), and therefore GLH is rightly projected.

COROLLARY II.

And for the same reason, if N be the centre of the circle G*g*HR, the centres of all circles passing through *g* and R, will be in the line *r*N*s* perpendicular to RS; so *n* is the centre of *gr*R. But then as *g*, R do not represent opposite points of the circle G*g*H, therefore all circles passing through *g*, R (as *gr*R), will be lesser circles, except G*g*HR.

PROP. VIII.

The extremities of the diameter of a projected circle are in the line of measures distant from the centre of the primitive, the semi-tangents of the least and greatest distances of the original circle from the pole of the primitive opposite to the projecting point.

Let EF (fig. 48.) be the original circle, IK the primitive (both seen edgewise), A the projecting point. The diameter of the projected circle is in the line of measures (*Cor.* 2. *prop.* 3. ; also the diameter EF is evidently projected into GH, of which the parts GC and CH are the semi-tangents of BE and BF. *Q. E. D.*

COROLLARY I.

The points where an inclined great circle cuts the line of measures within and without the primitive, are distant from the centre of the primitive, the tangent and co-tangent of half the complement of the circle's inclination to the primitive.

For CG (fig. 46.) = tangent of half EB, or of half the complement of IE, the inclination. And (because the angle EAF is right) CH is the co-tangent of GAC, or half EB.

COROLLARY II.

Hence the centre D (fig. 49.) of a projected circle is in the line of measures, distant from the centre of the primitive,

half the difference of the semi-tangents of its nearest and greatest distance from the opposite pole, if it encompasses that pole; but half the sum of the semi-tangents if it lie on one side of the pole of projection.

COROLLARY III.

And the radius is half the sum of the semi-tangents, if the circle encompasses the pole; or half the difference, if it lies on one side.

PROP. IX.

In any projected circle, the segments of the diameter, intercepted between its extremities and one projected pole, are to each other as the segments between the same extremities and the other.

Let EF (fig. 49.) denote the original circle, P and Q its poles, qH the plane of projection, A the projecting point. Then GH is the diameter of the projected circle, p and q the projected poles. Then since EP = PF, the angle GAH is bisected by the line Ap; hence (3. 6.) Gp : pH : : AG : AH.

Again, produce HA to L; then (13. 1.) HAG, GAL = two right angles = (31. 3.) 2pAq = 2pAG + 2GAq = HAG + 2GAq; hence GAL is bisected by Aq; therefore (A. 6.) Gq : qH : : AG : AH; consequently Gq : qH : : Gp : pH. *Q. E. D.*

PROP. X.

The projected poles of any circle are in the line of measures, within and without the primitive, and distant from its centre the tangent and co-tangent of half its inclination to the primitive.

The poles P, p (fig. 50.) of the circle EF are projected into D and d; and CD is the tangent of CAD or half BCP, that is, of half GCI, the inclination of the circle ICK, parallel to EF. Likewise Cd is the tangent of CAd, or the co-tangent of CAD. *Q. E. D.*

COROLLARY I.

The pole of the primitive is its centre; and the pole of a right circle is in the primitive.

COROLLARY II.

The projected centre * of any circle is always between the projected pole (nearest to it on the sphere) and the centre of the primitive; and the projected centres of all circles lie between the projected poles.

* The young student must observe, that the *projected centre* is the point on the plane of projection into which the centre of the original circle is projected; and is a different point from the centre of the projected circle; the latter always lying beyond the projected pole nearest to the centre of the primitive.

For the middle point of EF, or its centre, is projected into *s*; and all the points in P*p* (in which are all the centres) are projected into D*d*.

COROLLARY III.

If P (fig. 51.) be the projected centre of any circle EFG, any right lines EG, FH passing through P will intercept equal arches EF, GH.

For in any circle of the sphere, any two lines passing through the centre, intercept equal arches; and these are projected into right lines, passing through the projected centre P, and therefore EF, GH represent equal arches.

LEMMA.

If two planes pass through a globe, and also intersect each other, and, from the two points where their common section meets the surface, two circles equally distant from those points be described, the arcs of those circles, intercepted between the cutting planes, on the same side of their common section, are equal to each other.

Case 1. When the cutting planes pass through the centre of the sphere. Let ABDPLG and ACEP (fig. 52.) be the cutting planes, AP their common section, BC and DE the intercepted arcs, BF, CF, DH, EH the sections of their planes and the cutting planes; BC is equal to DE. Because ABDP and ACEP pass through the centre of the globe, AP is a diameter to each of the circles formed by the intersection of these planes and the spherical

surface. Because A and P are the poles of the circles BC and DE, the line AP is perpendicular to the planes of those circles (*definition* 2. *Spher. Trig.*); therefore AFB, AFC, PHD, PHE are right angles (*def.* 1. 2 *Supp.*); consequently FB is parallel to HD, and FC to HE (28. 1.); whence the angles BFC, DHE are equal (9. 2 *Supp.*) The lines BF, FC, DH, HE are also equal, being the sines of equal arcs; whence the arc BC is equal to DE (26. 3.)

Case. 2. When one of the planes passes through the centre of the globe, and the other does not.

Let ABDP (fig. 53.) be the circle formed by the intersection of the superficies of the sphere with the plane passing through the centre, ACEM the circle whose plane does not pass through the centre, AM the common section of the planes, O the centre of the circle ABDP; BF, DH the common sections of the planes BCF, DEH with the plane ABDP; join AO, MO, FC, LC, HE, NE, AC, ME; then, as in the first case, FB, FC, HD, HE are all equal; also AF = MH. But the angle FAL = HMN (5. 1.), and AFL = MHN, being both right; therefore AL = MN (26. 1.) Moreover, in the triangles ALC, EMN, the side AC = ME, and the angle CAL = EMN; hence LC = NE (4. 1.): then, in the triangles LFC, EHN, we have LF = NH, FC = HE, and LC = NE; therefore the angle LFC = NHE, consequently BC = DE.

Case 3. When neither of the planes passes through the centre of the sphere.

Through the common section of these planes and the centre of the sphere, let a third plane pass; then, by the last

case, the arches of one of the equidistant circles, intercepted between the third plane and each of the others, and the corresponding arcs of the other similarly intercepted arc, are equal, each to each; and therefore their sums or differences will be equal. But the sum of those arcs, when the given cutting planes fall on the opposite sides of the third plane, and their difference when they fall on the same side, will be the arcs intercepted between the first mentioned cutting planes.

PROP. XI.

If EFGH, efgh (fig. 51. and 54.) *represent two equal circles, whereof* EFGH *is as far from its pole* P, *as* efgh *is from the projecting point; any two right lines* EP, FP *drawn through* P *will intercept the representatives of equal arcs of those circles; on the same side, if* P *falls within the circles; but on the contrary side, if without; that is* EF = ef, *and* GH = gh.

For, by the Lemma, two planes passing through the pole of the original circle, represented by EFGH, and the projecting point, will cut off equal arcs of those circles, and these planes will intersect the plane of projection in right lines passing through the projected pole P.

COROLLARY I.

If a circle is projected into a right line EF (fig. 55.) perpendicular to the line of measures EG; and if from the centre of the primitive C, a circle *ef* P be described, passing through

P the pole of EF, and PF be drawn; then the arc $ef = EF$. And if any other circle be described whose vertex is P, the arc ef will always be equal to EF.

COROLLARY II.

If from the projected pole of a great circle there be drawn two right lines, the intercepted arc of that circle will be equal to the intercepted arc of the primitive.

COROLLARY III.

If from an angular point, two right lines be drawn through the poles of its sides, the intercepted arc of the primitive will be the measure of that angle.

For the arc of a great circle contained between the poles measures the inclination of the axes, or the inclination of the planes, of those circles.

PROP. XII.

If QH, NK (*fig.* 56. *and* 57.) *be two equal circles, whereof* NK *is as far from the projecting point as* QH *from its pole* P; *and if they be projected into the circles whose radii are* MC *or* CL, *and* DF *or* FG, F *being the centre of* DG, *and* E *the projected pole; I say, the pole* E *will be distant from their centres in proportion to the radii of the circles; that is,* CE : EF : : CL : DF or FG.

For, since NK and ML are parallel, and the arch $NI = PH$, therefore angle $ELI = NKI$ (or nKI) $= GIP$; there-

fore the triangles IEL and IEG are similar; whence EL : EI : : EI : EG.

Again, the angle EMI = KNI = PIQ, and therefore the triangles IEM and IED are similar; whence EM : EI : : EI : ED. Therefore $EI^2 = EL \times EG = EM \times ED$; consequently EM : EL : : EG : ED; and, by composition, $\frac{EM + EL}{2} : \frac{EM - EL}{2} :: \frac{EG + ED}{2} : \frac{EG - ED}{2}$; that is CM : EC : : FG : EF. *Q. E. D.*

COROLLARY I.

Hence, if the circle KN be as far from the projecting point as QH is from either of its poles, and if E, O be its projected poles; then will EL : EM : : ED : EG : : OD : OG.

This follows from the foregoing demonstration and prop. 9.

COROLLARY II.

Hence, also, if F be the centre, and FD the radius of any circle QH, and E, O the projected poles; then EF : DF : : DF : FO.

For it follows from Cor. 1. that $\frac{EG + ED}{2} : \frac{EG - ED}{2} :: \frac{OG + OD}{2} : \frac{OG - OD}{2}$. Whence, inversely, EF : DF : : DF : FO.

COROLLARY III.

Hence if the circle DBG (fig. 58.) be as far from its projected pole P, as LMN is from the projecting point; and if any right lines be drawn through P, as MPG, NPK, they will cut off similar arches GK, MN in the two circles.

For, from the centres C, F, draw the lines CN, FK; then, since the angles CPN and FPK are equal, and by this prop. CP : CN : : FP : FK; therefore (by 7. 6.) the triangles PCN and PFK are similar; and the angle PCN = angle PFK; therefore the arches MN and GK are similar.

COROLLARY IV.

Hence, also, if through the projected pole P of any circle DBG, a right line BPK be drawn; then, I say, the degrees in the arc GK shall be the measure of DB in the projection; and the degrees in DB, shall be the measure of GK in the projection.

For (by prop. 11.) the arc MN is the measure of DB, and therefore GK, which is similar to MN, will also be the measure of it.

COROLLARY V.

The centres of all projected circles are beyond the projected poles (in respect to the centre of the primitive); and none of their centres can fall between them.

COROLLARY VI.

Hence it follows (by cor. 5. and prop. 10. cor. 3.), that all circles that are not parallel to the primitive, have equal arches on the sphere represented by unequal arches on the plane of projection.

For if P (fig. 51.) be the projected centre, then GH is greater than EF.

SCHOLIUM.

It will be easy, by the foregoing propositions, to describe the representation of any circle, and the reverse will, easily show what circle of the sphere any projected circle represents. What follows hereafter is deduced from the foregoing propositions, as will be shown.

If the sphere was to be projected on any plane parallel to the primitive, it is all the same thing; for the cones of rays issuing from the projecting point, being all cut by parallel planes into similar sections, it only makes the projections greater or less, according to the distance of the plane of projection, whilst they are still similar; and it amounts to no more than projecting from different scales upon the same plane. And, therefore, the projecting of the sphere on the plane of a lesser circle is only projecting it upon the great circle parallel thereto, and continuing all the lines of the scheme to that lesser circle.

PROP. XIII. PROB.

To draw a circle parallel to the primitive at a given distance from its pole.

RULE.

Draw two diameters to the primitive AB, DE (fig. 59.), at right angles to each other. Set the distance of the parallel from the pole opposite the projecting point, on the primitive from D to F; join EF, cutting AB in I; from the centre C, with the radius CF, describe the circle IG, and the thing is done.

By the Plane Scale.

With the radius CI, equal to the circle's distance from the pole of the primitive opposite the projecting point, describe the required circle IG. Here it may be remarked, once for all, that, in all these projections by the plane scale, the radius of the primitive is equal the chord of 60° = tangent of 45° = semi-tangent of 90°.

The radius of the projected circle is the semi-tangent of the distance of its original circle from the pole of the primitive, by Cor. 3. prop. 8.

PROP. XIV. PROB.

To draw a lesser circle perpendicular to the primitive at a given distance from the pole of that circle.

RULE.

Through the pole B (fig. 60.) draw the line of measures AB; make BG equal to the circle's distance from its pole, and draw CG and GF perpendicular to it; with the radius FG describe the circle GI required.

By the Scale.

Set the secant of the circle's distance from its pole from C to F, which will give the centre. With the tangent of that distance for a radius, describe the circle GI. Or thus: make BG the circle's distance from its pole; and GF its tangent, set from G, gives F the centre; through G describe the circle GI from the centre F. (This construction is manifest from prop. 5.)

COROLLARY.

Hence a great circle perpendicular to the primitive, is a right line DCE drawn through the centre perpendicular to the line of measures. (See cor. 3. prop. 2.)

SCHOLIUM.

When the centre F lies at too great a distance, draw EG to cut AB in H; or lay the semi-tangent of DG from C to H; and through the three points G, H, I, draw a circle with a bow.

PROP. XV. PROB.

To describe an oblique circle at a given distance from a pole given.

RULE.

Draw the line of measures AB (fig. 61.) through the given point *p*, if that point is given; and draw DE perpendicular to it; also draw E*p*P. Or, if the point *p* is not given, set the height of the pole above the primitive from B to P. Then from P set off PH = PI = the circle's distance from its pole; and draw EH, EI to intersect AB in F and G. On FG, as a diameter, describe the circle required.

By the Scale.

Apply C*p* to the line of semi-tangents, and the distance of the pole of the circle, required to be drawn from the pole of the primitive, becomes known; whence the greatest and least distances of that circle from the pole of the primitive are easily obtained. Take the semi-tangents of these distances, and set them from C to F and from C to G, in the same or

in different directions, according as the extremities of the circle are situated on the same or on different sides of the pole opposite to the projecting point. (See cor. 2. prop. 2.)

SCHOLIUM.

This method is evidently applicable either to great circles or less; but the former are more readily described by the following problems.

PROP. XVI. PROB.

To describe, about a given pole, a great circle oblique to the primitive.

RULE.

Let F (fig. 61.) be the given pole; draw the line of measures AB, and, at right angles to it, the diameter ED; join EF, and produce it to H; make HP = HD; join EP, and let it (produced if necessary) cut AB in p; from the centre p, with the radius pE, describe the great circle ELD. Or, make HK a quadrant, join EK, and through ELD describe the circle required.

By the Scale.

Apply CF to the semi-tangents, and lay down Cp the tangent of the degrees in CF; from the centre p, with the secant of the same number of degrees as radius, describe the circle. Or, make CL the semi-tangent of the complement

of the degrees in CF; through DLE describe a circle, and the work is done.

The arc of a great circle intercepted between the pole of the oblique circle and that of the primitive, measures the inclination of those circles to each other; and this arc is projected into its semi-tangent (by cor. 2. prop. 2.) Also the distance of the centre of an oblique great circle from the centre of the primitive, is the tangent of their inclination (by prop. 6.); whence the first method is manifestly correct; the second appears equally so, from cor. 3. prop. 2.

PROP. XVII. PROB.

Through two given points to describe a great circle.

RULE.

Let A, B (fig. 62.) be the given points. Through one of them A draw the diameter OD, and EF at right angles to OD; join EA, and let it meet the primitive in H; draw the diameter HI; join EI, and let DO, EI (produced if necessary) meet in G; through the three points A, B, G describe the circle required.

By the Scale.

Draw the diameter DAO; apply CA to the semi-tangents; lay its supplement, taken also on the semi-tangents from C to G; through A, B, G describe the circle. Or, apply AC to the tangents, and lay the tangent of its complement from C to G.

For, since HEI or AEG is a right angle (31. 3.), the points A and G represent opposite points on the sphere; therefore all circles passing through A and G are great circles; also, CA and CG are evidently the semi-tangents of arcs, which together make a semicircle, or the tangents of arcs which compose a quadrant.

If one given point is in the centre of the primitive, the right circle passing through the other given point will be the one required.

PROP. XVIII. PROB.

About a given pole, to describe a circle through a given point.

RULE.

Let P (fig. 63.) be the given pole, and B the given point; through PB describe the great circle AD (by the last prob.); draw the line BF touching AD in B; join CP, and produce it to meet BF in F; from the centre F, at the distance BF, describe the circle BGH required.

Any circle passing through B, and having its pole in DA, will have its centre in BF (by 18. 3. and cor. to prop. 4.); the centre of a circle, whose pole is P, will likewise lie in CP produced (by cor. 2. prop. 3); therefore the point F is the centre of the required circle.

PROP. XIX. PROB.

To find the poles of a given projected circle FNG (fig. 61.)

RULE.

Through the centre of the given circle draw the line of measures AG, and, perpendicular to it, the diameter ED. Join E, F; E, G; and let them (produced if necessary) meet the primitive in H and I; bisect HI in P, join EP, meeting the line of measures in *p*: *p* is the internal pole. And if the diameter PQ be drawn, and EQ joined by a right line meeting the line of measures in *q*, *q* will be the external pole.

When the circle is a great one, the poles may be found more expeditiously thus: Let DLE be the great circle; draw the line of measures AB and diameter DE; join EL, and produce it to K; make KH, KR each equal to AD = 90°; draw EH, ER meeting the line of measures in F and S, the poles required.

By the Scale.

Apply CF, CG (fig. 61.) to the line of semi-tangents, and note the degrees; take half the sum or half the difference of these degrees, according as F and G lie on the same side or on opposite sides of C, and lay their semi-tangent from C to *p*, the internal pole required. Make C*q* the semi-tangent of the supplement of C*p*, and *q* is the external pole.

When the circle encompasses the pole of the primitive, observe to lay C*p* toward the most distant extremity G.

For a great circle as DLE, draw the line of measures AB. Apply CL to the semi-tangents, and lay its complement from C to F, and the degrees in CL increased by 90° from C to S (both taken from the semi-tangents); F and S are the projected poles required.

The reason of these constructions is easily seen from Cor. 1. prop. 8. and Cor. 2. prop. 2.

COROLLARY.

The poles of a right circle are in the primitive, distant 90° from the intersection of the primitive and right circle.

PROP. XX. PROB.

To draw a great circle making a given angle with the primitive at a given point D (fig. 61.)

RULE.

Draw the diameter DE, and AB at right angles thereto; lay twice the measure of the given angle from K to K; join EK, and let EK meet AB in L; from the centre L through D describe the circle D*p*E required.

By the Scale.

Draw the diameters DE, AB as before, and lay the tangent of the given inclination from C to L; L is the centre of the circle required.

These constructions are evident from prop. 6. and 20. 3. Elem.

COROLLARY I.

When the given angle is a right one, the right circle passing through the given point is the circle required.

COROLLARY II.

To describe an oblique circle making a given angle with a given oblique circle D*p*E, *at the point* D *in the primitive.*

Draw E*p*P, and set the degrees in the given angle from P to K; join EK, meeting AB in L; and through D, L, E describe the circle required.

PROP. XXI. PROB.

Through a given point P *to describe a circle making a given angle with the primitive.*

RULE.

Through the given point P (fig. 64.) draw the diameter AB, and DE at right angles to AB; set twice the given angle from A to K; join BK, cutting CD in G; from the centre C, through G, describe the circle GI, and from the centre P, with the radius BG, describe an arc cutting GI in F; from the centre F, through P, describe the circle MPL required.

By the Scale.

With the tangent of the given angle as radius and centre C describe the circle IGF; and from the centre P, with the secant of the same angle, describe an arc crossing it in F, the centre of the circle required.

This problem depends upon the same principles as the last.

COROLLARY.

If the circles described from C and P do not meet, the problem is impossible.

PROP. XXII. PROB.

To describe a great circle making a given angle with a given oblique great circle FPR (fig. 65.), *at a given point* P *in that circle.*

RULE.

Through the given point P, draw the diameter DE; find N the centre of the given circle, and draw NI perpendicular to DE; join PN, and make the angle NPL equal to the given one; from the centre L, through P, describe the circle HPK required.

Let FPR meet DE, produced, in T, then PT is the representative of a semicircle at right angles to the primitive; consequently any great circle which passes through P, must pass also through T; whence (by prop. 7.) the centres of all such circles will lie in IN; and the circles make the same angles with each other as their radii drawn to the point of intersection. (Cor. to prop. 4.)

PROP. XXIII. PROB.

Through a given point P (*fig.* 66.) *to describe a great circle making a given angle with a given great circle* DE.

RULE.

About the given point P as a pole, describe (by prop. 15.) the great circle FG; find I the pole of the given circle DE (by prop. 18.); and about the pole I, describe (by prop. 14.) the less circle HLK, at a distance equal to the measure of the given angle, intersecting FG in H; about the pole H, describe the great circle APB required.

The distance of the poles of two great circles measures the inclination of their planes; whence the construction is obvious.

COROLLARY.

When the given angle is a right one, the required circle must be described through P and I, by prop. 16.

PROP. XXIV. PROB.

To describe a great circle cutting two given great circles abd, ebf (*fig.* 67.) *at given angles.*

RULE.

Find the poles *s*, *r* of the two given circles (by prop. 18.); about these poles, at distances respectively equal to the measures of the given angles, describe two circles (by prop. 14.); the point *p*, where these circles cut each other, is the pole of the circle *moq* required.

COROLLARY.

To draw a right circle which shall make a given angle with a given oblique great circle *abd*, describe a less circle, at a distance from the pole of the oblique circle equal to the measure of the given angle; the point where this circle intersects the primitive is the pole of the right circle required.

PROP. XXV. PROB.

Through a given point Z (*fig.* 68.) *to describe a great circle which shall touch a given less circle* NMQ.

RULE.

Parallel to the given circle, and at the distance of 90° from it, describe the circle VTR (by prop. 12. 13. or 14.) About the given point Z, as a pole, describe the great circle TS (by prop. 15.), intersecting VTR in T; about T, as a pole, describe the circle ZMS required.

For, through T, and the pole P of the given circle, let the great circle TMP pass, meeting NMQ in M; then, by the construction, MT is a quadrant; therefore the circle ZS, whose pole is T, passes through M, and consequently touches the circle NMQ.

PROP. XXVI. PROB.

To lay any given number of degrees on a great circle, or to measure a given arc of it.

RULE.

Case 1. When the great circle is the primitive. Lay down the given degrees from the scale of chords; or measure the given arc by the scale of chords.

For the primitive is an original circle, and must therefore be measured as in common geometry.

Case 2. When the given circle is a right one, as AB (fig. 59.) Through C, the centre of the primitive, draw DE at right angles to AB, then D and E are the poles of AB. Then to lay any proposed number of degrees from C, towards A, lay them on the primitive from D to F, and draw EF, cutting AB in I; CI contains the number proposed; for, by cor. 2. prop. 11. CI and AI are measured by the arcs DF and AF respectively,

Or CI and AI may be laid down or measured, by the line of semi-tangents; observing to measure AI on the scale from 90° toward the beginning. (See cor. 2. prop. 2.)

Case 3. When the given circle is an oblique one. Let AFI (fig. 69.) be the primitive; find the internal pole P of the given circle DEH (by prop. 18.); lay the degrees proposed from A to F, and draw PA, PF intercepting DE the arc required. Or to measure a given arc DE, draw PEF, PDA; AF is the measure of DE.

Or, find the external pole *p* of the given circle; set the given number of degrees from I to K, and draw *p*I, *p*K, intercepting the arc DE required. Or to measure DE; through D and E draw *p*I, *p*K, then IK is the measure of DE. (See cor. 2. prop. 11.)

PROP. XXVII. PROB.

To set any proposed number of degrees on a less circle, or to measure a given arc of it.

RULE.

Case 1. When the less circle is parallel to the primitive.

Let ADBE (fig. 59.) be the primitive; IGL the parallel; lay the number of degrees proposed from B to H; join C, B, C, H, intercepting GL, the arc required. Or to measure GL, draw CGB, CLH; BH is the measure of GL.

Or GL may be laid down or measured by the line of chords on the sector, CG being made the radius.

Case 2. When the less circle is not parallel to the primitive. Let DEH (fig. 69.) be the less circle; find its internal pole, P (*by prop.* 18.); describe AFK parallel to the primitive, and as far from the projecting point as the circle DEH is from its internal pole P; set the given degrees from A to F (*by case* 1.), and draw PA, PF intersecting the given circle in D, E; then DE is the arc required. Or to measure DE, draw PDA, PEF; then AF is the measure of DE. (*See prop.* 11.)

PROP. XXVIII. PROB.

To measure a given angle.

RULE.

Find the poles of the containing sides (the nearest if the angle be acute, otherwise the furthest), and from the angular point, through these poles, draw right lines meeting the primitive; the intercepted arc of the primitive is the measure of the given angle. *Cor.* 3. *prop.* 11.

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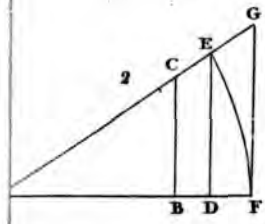

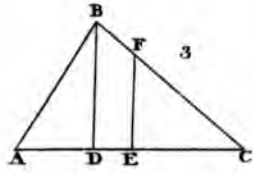

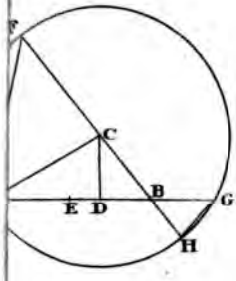

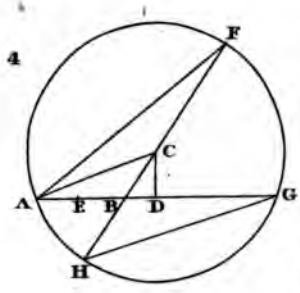

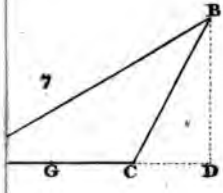

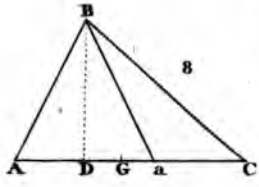

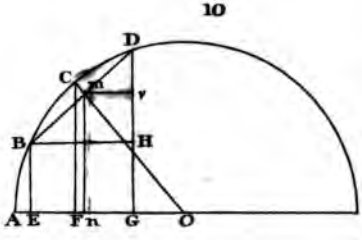

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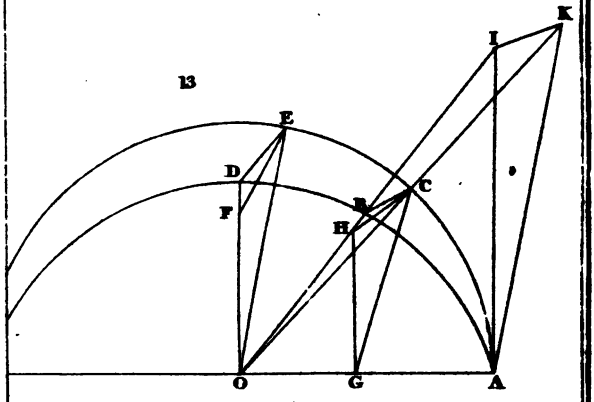

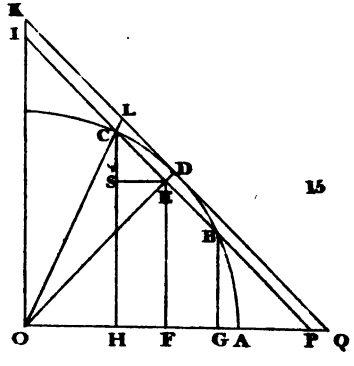

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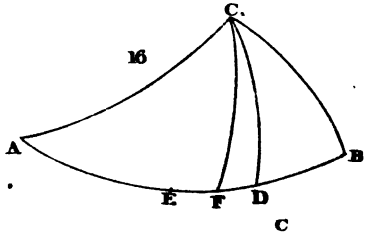

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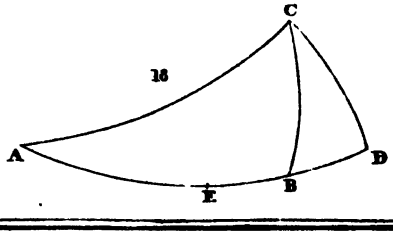

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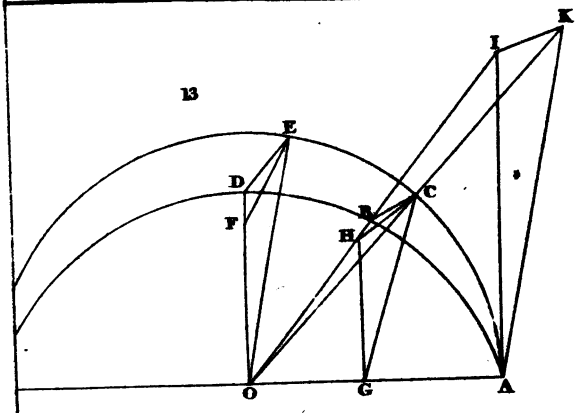

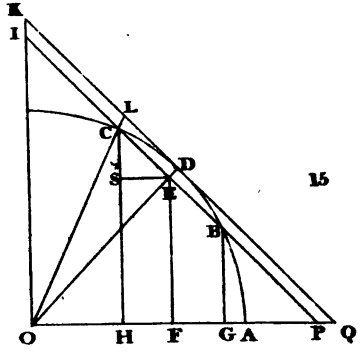

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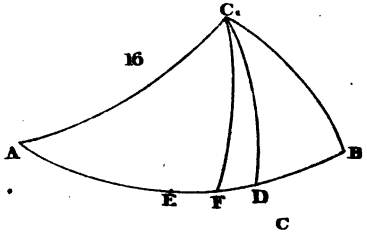

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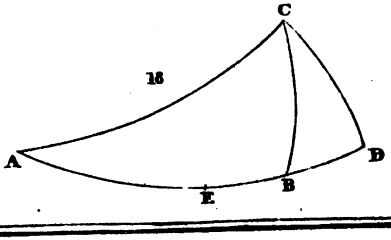

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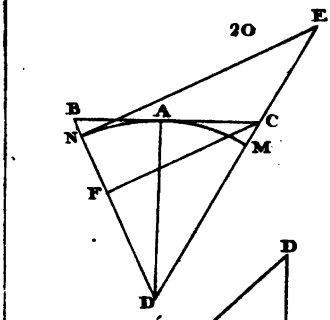

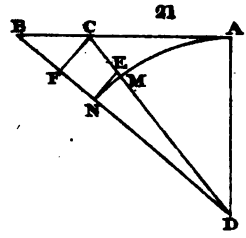

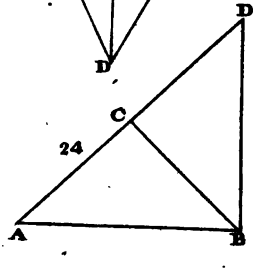

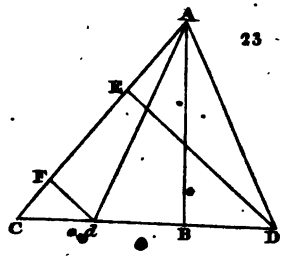

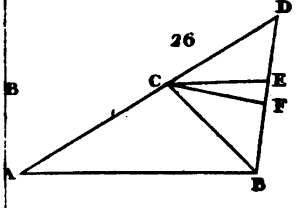

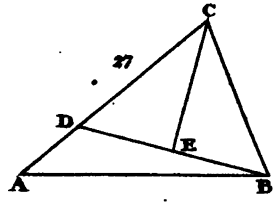

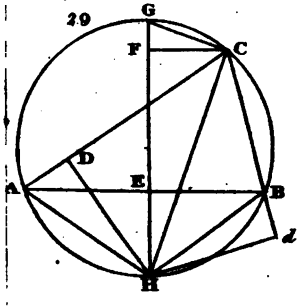

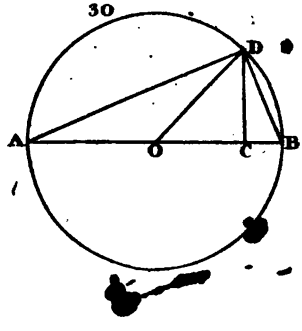

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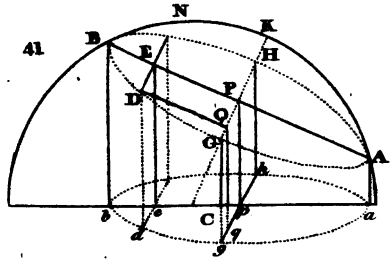
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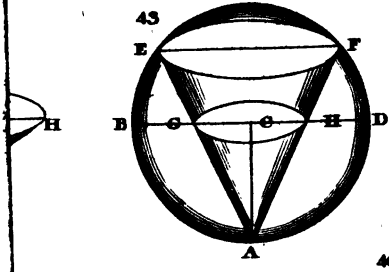
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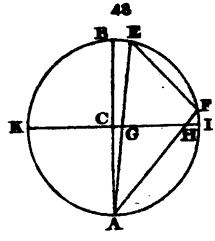
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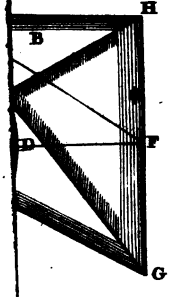
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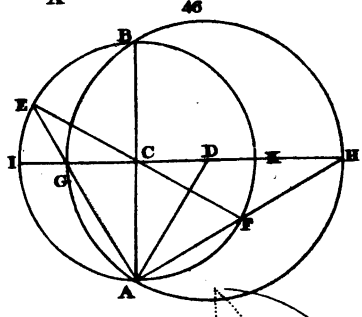
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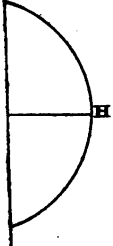
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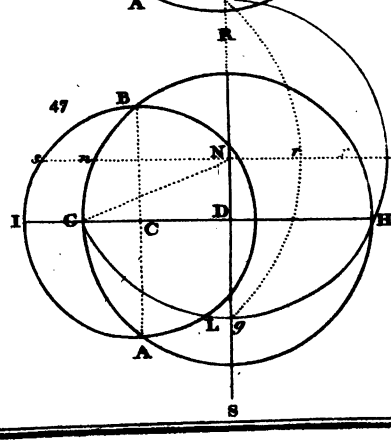
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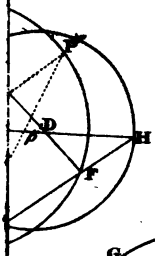

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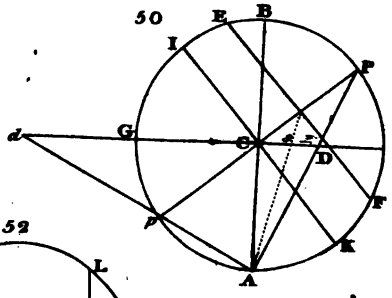

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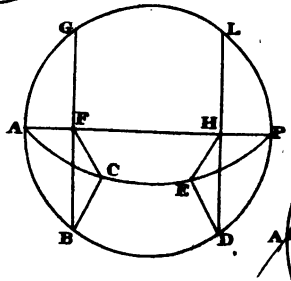

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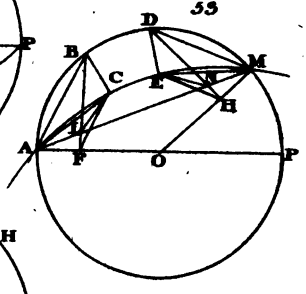

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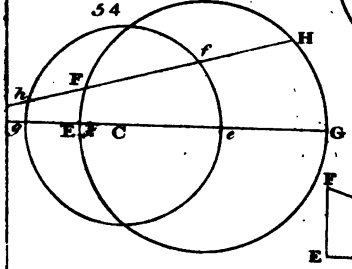

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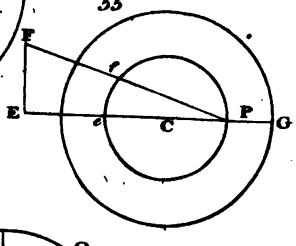

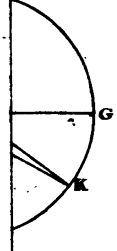

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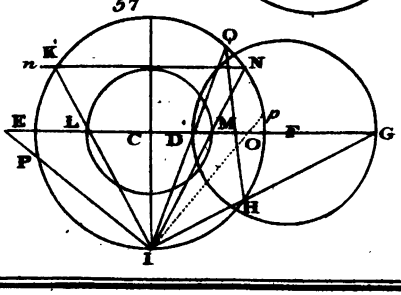

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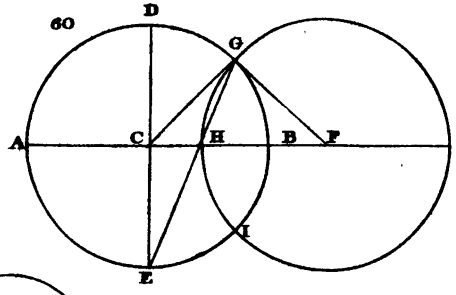

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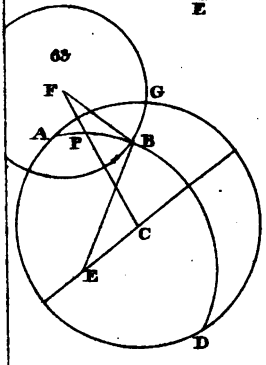

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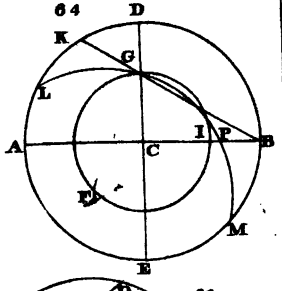

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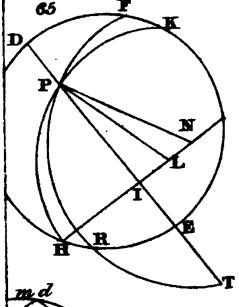

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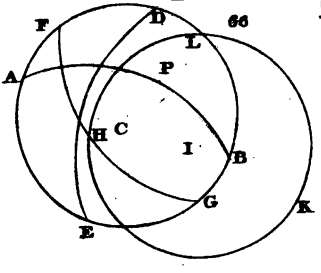

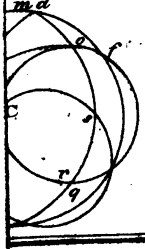

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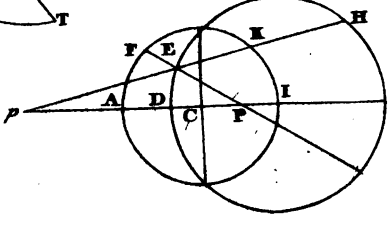

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